

MATHEMATICAL ESSAYS

ON

SEVERAL SUBJECTS:

CONTAINING

NEW IMPROVEMENTS and DISCOVERIES

IN THE

MATHEMATICS.

By the Rev. JOHN HELLINS.

———*Varias usus meditando extunderet artes*
Paulatim.—

VIRG.

*Ante omnia sufficit ad exhortationem studiorum, non capere id rerum
naturam, ut quicquid non est factum, ne fieri quidem possit.*

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P R E F A C E.

THE Public are here presented with a collection of papers, written, at different times, to amuse solitude, and with hopes of producing something which might be useful to the community.

A natural inclination pointed out the subject; and previous reading, as well as practice in the Mathematics, had made it probable to me, that some branches of those sciences were still capable of Improvement. For some years, therefore, I employed my leisure hours in trials to facilitate and extend those methods of computation, which have been long considered as the most useful. Of these trials several were attended with success; and the approbation which some specimens of these improvements have received from the learned* has been one of the strongest motives with me to publish this volume, as it afforded me the best

* See the Philosophical Transactions for 1780, and 1782.

evidence

P R E F A C E.

evidence I could possibly have of the utility of my inventions.

For the particulars of this book, I must refer the reader to the Essays themselves; because a detail of the contents would be much too long for this place.

As the printing of this work was undertaken by subscription, I most willingly embrace this opportunity of making my grateful acknowledgments to the subscribers, for the encouragement they have given me. And while I do this, I cannot but regret, that the work has been in the press above two years. The interruptions, however, which I met with in this Business, made that delay unavoidable. These interruptions were, indeed, very unexpected; and although they were produced by different immediate causes, yet they might all most justly be referred to those inconveniencies which are too commonly felt by the laborious part of the Clergy, and are too obvious to need any description.

*Green's-Norton, near Towcester,
Northamptonshire, March 25,
1788.*

J. HELLINS.

MATHEMATICAL

ESSAYS, &c.

ESSAY I.*

Of the Computation of Logarithms: containing some new Theorems for that purpose.

THE utility of logarithms is so well known, that much need not be said upon it. In our days he must be a slender mathematician who does not know that they are useful, not only in trigonometry, navigation, astronomy, the calculations of compound interest

* This Essay was read before the Royal Society March 16, 1780, and honoured with a place in the 70th Volume of their *Philosophical Transactions*. It is printed here without any material alteration, except the correction of some press errors, and the addition of the notes.

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and

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and annuities, but also in the finding of fluents, and the summation of infinite series.

Some of the greatest mathematicians that this kingdom ever produced, as Sir ISAAC NEWTON, Dr. HALLEY, Mr. COTES, and Mr. SIMPSON, have thought it not beneath them to improve the construction of logarithms; which strongly argues the utility of those artificial numbers, and may suggest to us, that the construction of them cannot be much further improved.

Now, although we should be very diffident in our expectations of improvement in any part of the mathematics after it has been handled by such great men, yet, if the method of computing be in general long and tedious, or if there still remain any particular difficulty, I believe, no good reason can be given why every attempt to abridge the one, or remove the other, should be discouraged. The easy method of computing the logarithms of large numbers given in page 49 of Mr. SIMPSON's pamphlet on Trigonometry is a proof, that those gentlemen*, who were of opinion that nothing better was to be hoped for, or expected, than what they published on the subject in the beginning of this century, were mistaken. And the following theorems, inferior to none as to con-

* See the *Construction of Logarithms* prefixed to Sherwin's Mathematical Tables.

vergency,

vergency, and useful in deducing the logarithms of great fractions from those of small ones, or the logarithms of small numbers from those of great ones, may be considered as another proof of the mistake before mentioned. I have only to add here, that these theorems are new to me; and if they are so to the public, I humbly presume they will be acceptable.

THEOREM I.

The log. of $\frac{p+q}{p} = 2 \times \log. \text{ of } \frac{2p+2q}{2p+q} + \log. \text{ of } \frac{(2p+q)^2}{(2p+p)^2 - qq}$.

DEMONSTRATION.

$$\begin{aligned} \frac{(2p+q)^2}{(2p+q)^2 - qq} \times \frac{(2p+2q)^2}{(2p+q)^2} &= \frac{1}{(2p+q)^2 - qq} \times \frac{(2p+2q)^2}{1} = \frac{4pp + 8pq + 4qq}{4pp + 4pq} = \\ \frac{pp + 2pq + qq}{p \times p + q} &= \frac{p+q}{p} : \text{consequently, } \log. \frac{(2p+q)^2}{(2p+q)^2 - qq} + 2 \log. \frac{2p+2q}{2p+q} \\ &= \log. \frac{p+q}{p}. \quad \text{Q. E. D.} \end{aligned}$$

COROLLARY.

If $q=1$, and we write n for p , the theorem becomes $\log. \frac{n+1}{n} = 2 \log. \frac{2n+2}{2n+1} + \log. \frac{(2n+1)^2}{(2n+1)^2 - 1}$, which expression perhaps is of more frequent use than that above.

THEOREM II.

$$\text{Log. } \frac{p+q}{p} = 2 \log. \frac{2p+q}{2p} - \log. \frac{\overline{2p+q}^2}{2p+q)^2 - qq}.$$

DEMONSTRATION.

$\frac{2p+2q}{2p+q} \times \frac{2p+q}{2p} = \frac{2p+2q}{2p} = \frac{p+q}{p}$: therefore, $\log. \frac{p+q}{p} = \log. \frac{2p+2q}{2p+q} + \log. \frac{2p+q}{2p}$. But it has been proved above, that $\log. \frac{p+q}{p} = 2 \log. \frac{2p+2q}{2p+q} + \log. \frac{\overline{2p+q}^2}{2p+q)^2 - qq}$. If now we take this equation from twice the last, there will remain $2 \log. \frac{p+q}{p} - \log. \frac{p+q}{p} = 2 \log. \frac{2p+2q}{2p+q} + 2 \log. \frac{2p+q}{2p} - 2 \log. \frac{2p+2q}{2p+q} - \log. \frac{\overline{2p+q}^2}{2p+q)^2 - qq}$: that is, $\log. \frac{p+q}{p} = 2 \log. \frac{2p+q}{2p} - \log. \frac{\overline{2p+q}^2}{2p+q)^2 - qq}$. Q. E. D.

COROLLARY.

Putting $q = 1$, and $n = p$, as above, we have

$$\log. \frac{n+1}{n} = 2 \log. \frac{2n+1}{2n} - \log. \frac{\overline{2n+1}^2}{2n+1)^2 - 1}.$$

I will now set down some examples of the use of these theorems, beginning with theorem 1.

The

The first example of the utility of this theorem may be in computing the logarithm of the number 2.

It is well known to mathematicians, that the computation of this logarithm was formerly a very laborious task: and although the work be much shortened by help of the converging series invented by the illustrious Sir ISAAC NEWTON, still the logarithm of 2 has not been directly computed without many figures by any theorem I have yet seen. The easiest computation of it that has come to my hands is in page 44 of the late ingenious Mr. THOMAS SIMPSON'S * pamphlet on Trigonometry and Logarithms. His series consists of the 1st, 3d, 5th, &c. powers of $\frac{1}{3}$.

If now we put $n=1$ in the theorem $(\log. \frac{n+1}{n} = 2 \log. \frac{2n+2}{2n+1} + \log. \frac{(2n+1)^2}{2n+1}^2 - 1)$ we shall have $\log. \frac{2}{1} = 2 \log. \frac{4}{3} + \log. \frac{9}{8}$. Here then the fractions, whose odd powers are to be used, are $\frac{4}{3}$ and $\frac{9}{8}$; consequently, in the series formed from $\frac{4}{3}$, about one half of the number of terms taken by Mr. SIMPSON will give the result true to as many places of figures as his; and, from the fraction $\frac{9}{8}$, much fewer terms will suffice. To show how fast these series converge, I will set down, of each, terms enough to give the logarithm of 2 true to ten places of figures.

* The same series may be found in the learned Dr. Saunderson's Algebra.

The

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The odd powers of $\frac{1}{7}$ divided by their respective indices.

1st,	0.14285714286
3d,	0.00097181730
5th,	0.00001189980
7th,	0.00000017347
9th,	0.00000000275
11th,	0.00000000004

The sum, 0.14384103622 is $\frac{1}{4}$ l. of $\frac{1}{7}$.

0.57536414488 twice l. $\frac{1}{7}$.

The odd powers of $\frac{1}{17}$ divided by their respective indices.

1st,	0.05882352941
3d,	0.00006784721
5th,	0.00000014086
7th,	0.00000000035

The sum, 0.05889151783 is $\frac{1}{2}$ l. of $\frac{1}{17}$.

Log. $\frac{1}{7}$, 0.11778303566
 $2 \log. \frac{1}{7}$, 0.57536414488

Log. of 2. 0.69314718054

Note. It is evident to an intelligent reader, that the multiplication by 4, as well as that by 2, may always be avoided. And it will appear a little further on, that, swift as the convergency of the series formed from $\frac{1}{17}$ is, yet it is the slowest that need be used in computing a Table of Hyperbolic Logarithms.

But

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But it is obvious, that this operation gives not only the logarithm of 2 but that of 3 also: for the logarithm of 4 being given from that of 2, and the logarithm of $\frac{4}{3}$ computed above, the logarithm of 3 is had, being = log. of 4 — log. of $\frac{4}{3}$.

$$\text{Log. of 4} \quad 1.38629436108$$

$$\text{Log. of } \frac{4}{3} \quad 0.28768207244$$

$$\text{Log. of 3} \quad 1.09861228864$$

Other examples of the use of these theorems in shewing how easily the logarithms of great fractions are derived from those of small ones.

The logarithm of $\frac{64}{63}$ being given, or computed, we may very easily find the logarithm of $\frac{32}{31}$: for, by theorem 1. $2 \log. \frac{64}{63} + \log. \frac{63^2}{63^2 - 1} = \log. \frac{32}{31}$. Here the fraction, whose odd powers are to be used in the series, is $\frac{1}{7937}$; and the very first term of it will give the logarithm true to twelve places of figures.

Again, if the logarithm of $\frac{16}{15}$ were to be computed from that of $\frac{32}{31}$ found above, we should have $2 \log. \frac{32}{31} + \log. \frac{31^2}{31^2 - 1} = \log. \frac{16}{15}$. Here the fraction to be used in the series is $\frac{1}{7921}$, the first term of which will give the logarithm true to ten places of figures.

In

In like manner, from the logarithm of $\frac{16}{15}$ we may find that of $\frac{8}{7}$; from logarithm of $\frac{8}{7}$ that of $\frac{4}{3}$; and from the logarithm of $\frac{4}{3}$ that of $\frac{2}{3}$, as is done above. The respective fractions for the series will be $\frac{1}{449}$, $\frac{1}{57}$, and $\frac{1}{17}$.*

Thus far the fractions I have taken have even numbers for their numerators; let us now take one whose numerator is an odd number, $\frac{9}{8}$. Here n being $= 3\frac{1}{2}$, $\log. \frac{9}{7} \left(\frac{4\frac{1}{2}}{3\frac{1}{2}} \right) = 2 \log. \frac{9}{8} + \log. \frac{64}{3}$; and the fraction whose odd powers are to be used is $\frac{1}{127}$. Hence we have the log. of $\frac{8}{7}$ (for $\frac{9}{7} \div \frac{9}{8} = \frac{8}{7}$), and may proceed to find the logarithm of 2 as above. But the logarithm of $\frac{8}{7}$ may be directly derived from the equation thus: the equation in other terms is, $\log. 9 - \log. 7 = 2 \log. 9 - 2 \log. 8 + \log. \frac{64}{3}$; then, by transposition, $\log. 8. - \log. 7 = \log. 9 - \log. 8 + \log. \frac{64}{3}$, or $\log. \frac{8}{7} = \log. \frac{9}{8} + \log. \frac{64}{3}$.

But when the numerator of the fraction whose logarithm is given is odd, theorem 2^d. is more commodious. For, taking $\frac{9}{8}$, as before, we have $2 \log. \frac{9}{8} - \log. \frac{81}{8} = \log. \frac{5}{4}$, where the fraction to be involved is $\frac{1}{64}$. Again, $2 \log. \frac{5}{4} - \log. \frac{25}{4} = \log. \frac{3}{2}$, where the fraction is $\frac{1}{9}$. And $2 \log. \frac{3}{2} - \log. \frac{9}{2} = \log. \frac{2}{3}$, where we have only to take the difference of logarithms, as the logarithm of $\frac{9}{8}$ as well as that of $\frac{3}{2}$ is given.

* See page 6.

All the above calculations are of hyperbolic logarithms; but the same theorems hold good for Mr. BRIGGS's, or any other. I will give an example in the computation of BRIGGS's logarithm of 7 from others already known.

Let the logarithms of 100, 99, and 50, be given; then (by theorem 1.) $2\log. \frac{100}{99} + \log. \frac{99^2}{99^2-1} = \log. \frac{50}{49}$, or, $\log. \frac{100}{99} + \frac{1}{2} \log. \frac{99^2}{99^2-1} = \frac{1}{2} \log. \frac{50}{49}$; and then $\frac{1}{2} \log. 50 - \frac{1}{2} \log. \frac{50}{49} = \frac{1}{2} \log. 49 = \log. 7$.

Log. of $\frac{100}{99}$	-	-	-	0.00436480540245
$\frac{1}{2} \log. \text{ of } \frac{99^2}{99^2-1}$	$\left(= \frac{0.43429 \text{ etc.}}{19601} = \right)$			0.00002215675128
$\frac{1}{2} \log. \text{ of } \frac{50}{49}$	-	-	-	0.00438696215373
$\frac{1}{2} \log. \text{ of } 50$	-	-	-	0.84948500216801
Log. of 7	-	-	-	0.84509804001428

S C H O L I U M.

Neither the number 2, nor the fraction $\frac{64}{63}$ is chosen as the most advantageous to begin with in computing a table of logarithms; but they are taken as some of the first that occurred, to show the use of these theorems. Perhaps there are other instances in which they might be shown to much more advan-

tage; but I hope their use will appear from the few examples given. They may indeed be transformed so as to be more commodious in particular cases, and there may be some others derived from them, one or two of which I will here put down.

It is evident from theorem 1. and 2. that $2 \log. \frac{2p+2q}{2p+q}$
 $+ \log. \frac{\overline{2p+q}^2}{\overline{2p+q}^2 - qq} = 2 \log. \frac{2p+q}{2p} - \log. \frac{\overline{2p+q}^2}{\overline{2p+q}^2 - qq}$; conse-
 quently, $2 \log. \frac{2p+2q}{2p+q} + 2 \log. \frac{\overline{2p+q}^2}{\overline{2p+q}^2 - qq} = 2 \log. \frac{2p+q}{2p}$, or,
 $\log. \frac{2p+q}{2p} = \log. \frac{2p+2q}{2p+q} + \log. \frac{\overline{2p+q}^2}{\overline{2p+q}^2 - qq}$, which may
 be called theorem 3.

Again, this equation may be thus expressed: $\log. \frac{2p+q}{2p+q} - \log. 2p = \log. 2p + 2q - \log. 2p + q + \log. \frac{\overline{2p+q}^2}{\overline{2p+q}^2 - qq}$;
 and, by transposition, $2 \log. 2p + q = \log. 2p + 2q + \log. 2p + \log. \frac{\overline{2p+q}^2}{\overline{2p+q}^2 - qq}$, which may be called theorem 4.

And this is, in effect, one of the theorems given by Dr. HALLEY, in the Philosophical Transactions, No. 216, and of which, on account of its quick convergency, the doctor said, that, in his opinion, nothing better was to be hoped.

There

There are yet some contrivances, different from those mentioned in the beginning of SHERWIN's book of mathematical tables, or any other that have come to my hands, whereby the labour of computing a table of logarithms is shortened; but to explain them would require more time than my present situation affords me.

The observations and reasonings which led me to the discovery of the above theorems, I imagine, need not here be mentioned. Such as they are, I beg leave to lay them before the candid and skilful in these matters, in hopes that the invention will appear to them, as it does to me, a useful one.

It has, indeed, been objected, by a gentleman of my acquaintance, that improvements in the construction of logarithms cannot now be useful, because logarithms are already constructed.

I answer, That argument, if it has any weight, operates equally against Sir ISAAC NEWTON, Dr. HALLEY, Mr. COTES, Mr. SIMPSON, and several other ingenious mathematicians; for logarithms were invented, and tables of them constructed, before their time. So that, if I should be thought to have misemployed my time in improving the computation of these artificial numbers, I have some consolation

in thinking, that I have therein followed the example of the very respectable company just mentioned.

I trust, however, that, with mathematicians, every improvement in calculation will be acceptable.

E S S A Y II.

On the Computation of Logarithms: containing several new Theorems for that purpose, and a new method of constructing a Table of those artificial Numbers.

THE substance of this essay is taken from papers of which a hint was given towards the end of the preceding one. The theorems were selected from a considerable number, where it was not easy to determine which were the most useful. All the series here given converge very swiftly, but some of them about twice as fast as any other, that I know of, yet published; the approximations to the values of those series have different degrees of accuracy, each of which is ascertained; and the method, thence derived, of computing, examining, or enlarging a table of logarithms, is very easy.

T H E O R E M I.

If there be four numbers, A, B, C, and D, in a decreasing arithmetical progression, whose common difference is 1, that is, if $A - 1 = B$, $B - 1 = C$, and $C - 1 = D$, and if you put $C = a$, $\frac{a+2}{2a+1} = p$, the modulus of any kind of logarithms

$= M,$

$=M$, and $M \times$ the series $\frac{1}{pa^3} + \frac{1}{2p^2a^6} + \frac{1}{3p^3a^9}$, &c. $= S$;
 then shall $S = 3 \log. C + \log. A - 3 \log. B - \log. D$. Or,
 using the above notation,

THEOREM II.

$2M \times$ the series $\frac{1}{2pa^3-1} + \frac{1}{3 \times \frac{2pa^3-1}{2}} + \frac{1}{5 \times \frac{2pa^3-1}{2}}$
 &c. $= S$, $= 3 \log. C + \log. A - 3 \log. B - \log. D$.

The demonstration of these two theorems being contained in the investigation of the two that follow, it may be omitted here.

EXAMPLE. Let the four numbers be 4, 3, 2, and 1; then, $C = a = 2$, $\frac{a+2}{2a+1} = \frac{2+2}{4+1} = \frac{4}{5} = 0.8$, and $pa^3 = 6.4$;
 and by theorem 1, we have $M \times: \frac{1}{6.4} + \frac{1}{2 \times 6.4^2} + \frac{1}{3 \times 6.4^3}$
 &c. $= S$: which series has a degree of convergency, that, for most other purposes, would not be considered as contemptible. But, by theorem 2, we have $2pa^3 - 1 = 12.8 - 1 = 11.8$, and $S =$

$$\begin{array}{r}
 M \times : \quad 0.16949152542 \\
 \quad + 0.00040575392 \\
 \quad + 0.00000174845 \\
 \quad + 0.00000000897 \\
 \quad + 0.00000000005
 \end{array}
 \left. \vphantom{\begin{array}{r} \\ \\ \\ \\ \end{array}} \right\} = 0.16989903681 \quad M =$$

3 log.

$3 \log. 2 + \log. 4 - 3 \log. 3 - \log. 1$; or, because $\log. 4 = 2 \log. 2$, and $\log. 1 = 0$, we have $5 \log. 2 - 3 \log. 3 = S$. If now we put $M=1$, and take the hyp. log. of $\frac{4}{3}$, computed in Essay I. page 7, $= 0.28768207244$, and call it b , we shall have $2 \log. 2 (= \log. 4) = b + \log. 3$, or $\log. 3 = 2 \log. 2 - b$, which being written for it in the equation above, we have $S = -\log. 2 + 3b$, or $\log. 2 = 3b - S$.

$$\begin{array}{r} 3b = 0.86304621732 \\ S = 0.16989903681 \\ \hline \end{array}$$

$$\text{Hyp. log. of } 2 = 0.69314718051$$

It is obvious that the last series, though it converges pretty fast, yet does not converge so quick as one in page 6, nor is the calculation of the hyp. log. of 2 by it so easy as by the series derived from $\frac{1}{17}$, not only because 17 is greater than 11.8, but because the divisions by the former, consisting of fewer figures, are more easily performed. But the proper use of these theorems is not to find the logarithms of small numbers, but those of great ones from the logarithms of others next them.

The value of p in the above theorems, it is evident, while the progression is of any whole positive numbers, cannot be greater than $\frac{1}{2}$, nor so little as $\frac{1}{4}$; for it will

will always be $= \frac{1}{2} + \frac{3}{4a+2}$: consequently, the greater a is, the faster will the series converge.

EXAMPLE II. Let A, B, C , and D be 102, 101, 100, and 99; then, a being = 100, we have $\frac{a+2}{2a+1} = p = \frac{102}{201} = 0.507462686567164$ &c. and $pa^3 = 507462.686567164$ &c. and by theorem 1, $S =$

$$\begin{aligned} M \times: & 0.000001970588235294 \\ & + 0.0000000000001941609 \\ & + 0.0000000000000000002 \end{aligned} \left. \vphantom{\begin{aligned} M \times: & 0.000001970588235294 \\ & + 0.0000000000001941609 \\ & + 0.0000000000000000002 \end{aligned}} \right\} = 0.000001970590176905 M$$

By theorem 2, we have the same value of S from the very first term of the series, which is $M \times \frac{402}{203999799} = 0.000001970590176905 M$.

THEOREM III.

If there be five numbers, A, B, C, D , and E , in arithmetical progression, whose common difference is 1, that is, $A-1=B, B-1=C$, &c. and if you put $C=a, \frac{a^4-4a^2}{6a^4-4a^2+1} = \pi$,

the modulus of any kind of logarithms = M , and $M \times$ the series

$$\frac{1}{\pi a^4+1} + \frac{1}{2 \times \pi a^4+1}^2 + \frac{1}{3 \times \pi a^4+1}^3, \text{ \&c.} = \Sigma; \text{ then shall } \Sigma =$$

$$4 \log. B + 4 \log. D - 6 \log. C - \log. A - \log. E. \quad \text{Or,}$$

THEOREM

THEOREM IV.

Retaining the notation used in the last theorem, $2M$
 $\times$ the series $\frac{1}{2\pi a^4 + 1} + \frac{1}{3 \times 2\pi a^4 + 1}^3 + \frac{1}{5 \times 2\pi a^4 + 1}^5$ &c. =
 $\Sigma, = 4 \log. B + 4 \log. D - 6 \log. C - \log. A - \log. E.$

EXAMPLE. Let $A, B, C, D,$ and $E,$ be 5, 4, 3, 2, and 1 respectively; then, $a = 3, \frac{a^4 - 4a^2}{6a^4 - 4a^2 + 1} = \pi$
 $= \frac{81 - 36}{486 - 36 + 1} = \frac{45}{451} = 0.09977827051$ very nearly, $\pi a^4 + 1$
 $= 9.08203991131,$ and, by theorem 3, $\Sigma = M \times :$
 $\frac{1}{9.08203991131} + \frac{1}{2 \times 9.08203991131}^2$ &c. Or, by theorem 4.
 $\Sigma = 2M \times : \frac{1}{17.16407982262} + \frac{1}{3 \times 17.16407982262}^3$ &c. which
series converges so fast, that, for most other purposes, it would be a desirable one; but, for the reason given in page 15, it is not so eligible as the series formed from $\frac{1}{17}.$

The value of $\pi,$ it is evident, while the progression is of any whole positive numbers, cannot be less than $\frac{45}{451}, (= \frac{1}{10} - \frac{1}{4510},)$ nor so great as $\frac{1}{6};$ for it will always be $= \frac{1}{6} - \frac{20a^2 + 1}{36a^4 - 24a^2 + 6} :$ and therefore, when a is a large number, as in the proper use of these theorems

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it

it always will be, the series will converge very swiftly.

EXAMPLE II. Let the five numbers be 102, 101, 100, 99, and 98; then, a being = 100, $\frac{a^4 - 4a^2}{6a^4 - 4a^2 + 1}$
 $= \pi$, is $= \frac{100000000 - 40000}{600000000 - 40000 + 1} = \frac{99960000}{599960001} = 0.166611107129$
 &c. and $\pi a^4 + 1 = 16661111.7129$ &c. and, by theorem 3, $\Sigma =$

$$M \times \left. \begin{array}{l} 0.0000000060020004501 \\ + 0.0000000000000001801 \end{array} \right\} 0.0000000060020006302M.$$

By theorem 4 we have, from the first term of the series,

$$M \times \frac{1199920002}{19992000599960001} = M \times 0.000000006002000630204 \text{ \&c.}$$

To find to how many places of figures Σ is given by the first term of the series, in theorem 4, when a is 100, call $\pi \frac{1}{6}$; then, $\frac{1}{24\pi^3} = \frac{216}{24} = 9$, and $\frac{9}{a^{12}} =$

$\frac{9}{1000000000000000000000000}$ for an approximate value of the second term: consequently, the logarithm obtained from the first term of this series, when $a = 100$, will be true to twenty-three places of figures.

And hence it is evident, that, if a had been 1000, the logarithm computed by the first term of the series would have been true to thirty-five places of figures; if a had been 10000, the logarithm would have been obtained to forty-seven places.

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In like manner may the accuracy of any term of any of the above series be ascertained. It appears in the calculations of the second example of the use of theorems 1 and 2, that, when a is 100, the value of S given by $\frac{2M}{2pa^3-1}$, the first term of the series in theorem 2, is true to eighteen places of figures; and if a had been 1000, which perhaps is as small a number as need be used in computing a table of logarithms, after the modulus is found, the same term would have given S true to twenty-seven places of figures.

Now, as the logarithms obtained from the first terms of these series are true to so many places of figures, we may find an approximate value of each of the coefficients, p and π , and thence derive other theorems, which shall be exact enough for the common cases, but shorter and more easy in their application.

The value of $p = \frac{a+2}{2a+1}$, in a series, is $\frac{1}{2} + \frac{3}{4a} - \frac{3}{8a^2} + \frac{3}{16a^3}$ &c.; consequently $2pa^3 - 1 = a^3 + \frac{3}{2}a^2 - \frac{3}{4}a + \frac{3}{8} - 1 = a^3 + \frac{3}{2}a^2 - \frac{3}{4}a - \frac{5}{8}$ very nearly, and $\frac{2M}{2pa^3-1} = \frac{2M}{a^3 + \frac{3}{2}a^2 - \frac{3}{4}a - \frac{5}{8}}$ very nearly, which, for ease in calculation, may be reduced to $\frac{4M}{2a^3 + 3a^2 - 1.5a - 1.25} = S$ nearly, (the difference from the truth being less than

D 2
M

$\frac{M}{2a^7}$, as will be shown hereafter,) which may be called THEOREM V.

EXAMPLE. Let the progression be, as above, 102, 101, 100, and 99, and $M=1$; then, we have $S = \frac{4}{2000000 + 30000 - 150 - 1.25} = \frac{4}{2029848.75} = 0.000001970590173$ &c. erring only in the fifteenth place of decimals. And if the numbers were 1002, 1001, 1000, and 999, the value of S , computed by this theorem, would be true to twenty-one places of figures.

The value of π also, expressed in series, is $\frac{1}{6} - \frac{10}{18a^2} - \frac{43}{108a^4}$ &c. and therefore, $2\pi a^4 + 1 = \frac{1}{3}a^4 - \frac{10}{9}a^2 - \frac{43}{54} + 1$ very nearly, $= \frac{1}{3}a^4 - \frac{10}{9}a^2 + \frac{11}{54}$, and $\frac{2M}{2\pi a^4 + 1} = \frac{2M}{\frac{1}{3}a^4 - \frac{10}{9}a^2 + \frac{11}{54}}$ nearly, or, for ease in calculating, $\frac{18M}{3a^4 - 10a^2 + 1.8\frac{1}{3}} = \Sigma$ nearly; (the error being less than $\frac{7M}{a^{10}}$;) and this may be called THEOREM VI.

EXAMPLE. Let the progression be 102, 101, 100, 99, and 98; then we have $\Sigma = \frac{18M}{300000000 - 100000 + 1.8333}$ &c. $= \frac{18M}{299900001.8333}$ &c. $= M \times 0.0000000600200063019$ &c. erring in the twentieth place of decimals.

EXAMPLE

EXAMPLE II. Let the five numbers, A , B , C , D , and E , be 1002, 1001, 1000, 999, and 998, and let $M=1$, to find Σ true to fourteen places.

$$\text{We have } \Sigma = \frac{18}{2999|990000001 \cdot 8\frac{1}{3}} = 0.000000000006000,$$

where the work is very short, as all the figures to the right hand of the perpendicular line in the divisor may be considered as cyphers; or indeed we may use only $3a^4$, the first term of the divisor in theorem 6.

And hence it is evident, that, had a been = 5000, or upwards, and $M=0.43429$ &c. the modulus of BRIGGS's logarithms, and the result wanted only to fourteen places, Σ might have been put = 0, and we should have had this equation, $\Sigma=0=4 \log. B + 4 \log. D - 6 \log. C - \log. A - \log. E$, where it is very evident, that, any four of these logarithms being given, the fifth may be found by a simple equation.

But what renders these theorems the more valuable is, that S and Σ are the third and fourth differences of logarithms, as will appear in the following

INVESTIGATION.

Let there be five numbers A , B , C , D , and E , in arithmetical progression, whose common difference is 1, that is, let $A-1=B$, $B-1=C$, &c. and call

call C, a ; then, by the nature of logarithms we have

First Differences of Logarithms.

Second Differences of Logarithms.

$$\begin{array}{l|l} \text{Log. } \frac{A}{B} = \log. A - \log. B & 2 \log. B - \log. A - \log. C = \log. \frac{B^2}{AC} \\ \text{Log. } \frac{B}{C} = \log. B - \log. C & 2 \log. C - \log. B - \log. D = \log. \frac{C^2}{BD} \\ \text{Log. } \frac{C}{D} = \log. C - \log. D & 2 \log. D - \log. C - \log. E = \log. \frac{D^2}{CE} \\ \text{Log. } \frac{D}{E} = \log. D - \log. E & \end{array}$$

Third Differences of Logarithms.

$$\begin{array}{l} 3 \log. C + \log. A - 3 \log. B - \log. D = \log. \frac{C^3 A}{B^3 D} \\ 3 \log. D + \log. B - 3 \log. C - \log. E = \log. \frac{D^3 B}{C^3 E} \end{array}$$

Fourth Difference of Logarithms.

$$4 \log. B + 4 \log. D - 6 \log. C - \log. A - \log. E = \log. \frac{B^4 D^4}{C^6 A E}.$$

And by writing $a+2$ for A , $a+1$ for B , a for C , &c. we have

First Differences of Logarithms.

Second Differences of Logarithms.

$$\begin{array}{l|l} \text{Log. } A - \log. B = \log. \frac{a+2}{a+1} & 2 \log. B - \log. A - \log. C = \log. \frac{(a+1)^2}{a \times a+2} \\ \text{Log. } B - \log. C = \log. \frac{a+1}{a} & 2 \log. C - \log. B - \log. D = \log. \frac{a a}{a a-1} \\ \text{Log. } C - \log. D = \log. \frac{a}{a-1} & 2 \log. D - \log. C - \log. E = \log. \frac{(a-1)^2}{a \times a-2} \\ \text{Log. } D - \log. E = \log. \frac{a-1}{a-2} & \end{array}$$

Third

Third Differences of Logarithms.

$$3 \log. C + \log. A - 3 \log. B - \log. D = \log. \frac{a^3 \times \overline{a+2}}{\overline{a+1}^3 \times \overline{a-1}}$$

$$3 \log. D + \log. B - 3 \log. C - \log. E = \log. \frac{\overline{a-1}^3 \times \overline{a+1}}{a^3 \times \overline{a-2}}$$

Fourth Difference of Logarithms.

$$4 \log. B + 4 \log. D - 6 \log. C - \log. A - \log. E = \log. \frac{\overline{a+1}^4 \times \overline{a-1}^4}{a^6 \times \overline{a+2} \times \overline{a-2}}.$$

In the above table it appears, that the first of the third differences of logarithms is $3 \log. C + \log. A - 3 \log. B - \log. D = \log. \frac{a^3 \times \overline{a+2}}{\overline{a+1}^3 \times \overline{a-1}}$. Now, $\frac{a^3 \times \overline{a+2}}{\overline{a+1}^3 \times \overline{a-1}}$ is $= \frac{a^4 + 2a^3}{a^4 + 2a^3 - 2a - 1}$, which, by dividing both numerator and denominator by $2a+1$, becomes $a^3 \times \frac{a+2}{2a+1} \div \overline{a^3 \times \frac{a+2}{2a+1} - 1}$; and if we put $\frac{a+2}{2a+1} = p$, and the modulus of any kind of logarithms $= M$, we shall have $3 \log. C + \log. A - 3 \log. B - \log. D = \log. \frac{a^3 \times \overline{a+2}}{\overline{a+1}^3 \times \overline{a-1}} = \log. \frac{pa^3}{pa^3 - 1}$, in series, $= M \times : \frac{1}{pa^3} + \frac{1}{2 \cdot pa^3} + \frac{1}{3 \cdot pa^3} \&c.$ which is THEOREM I.

But

But the logarithm of $\frac{pa^3}{pa^3-1}$ is known to be also =

$$M \times : \frac{2}{2pa^3-1} + \frac{2}{3 \cdot 2pa^3-1)^3} + \frac{2}{5 \cdot 2pa^3-1)^5} \&c. \quad \text{Conse-}$$

quently, we have $3 \log. C + \log. A - 3 \log. B -$

$$\log. D = 2 M \times : \frac{1}{2pa^3-1} + \frac{1}{3 \cdot 2pa^3-1)^3} + \frac{1}{5 \cdot 2pa^3-1)^5}$$

&c. which is THEOREM II.

In the above scheme it also appears, that the fourth difference of logarithms, $4 \log. B + 4 \log. D - 6 \log. C - \log. A - \log. E$, is = $\log. \frac{(a+1)^4 \times (a-1)^4}{a^6 \times a+2 \times a-2}$. But

$$\frac{(a+1)^4 \times (a-1)^4}{a^6 \times a+2 \times a-2} \text{ is } = \frac{a^8 - 4a^6 + 6a^4 - 4a^2 + 1}{a^8 - 4a^6} = a^4 \times \frac{a^4 - 4a^2}{6a^4 - 4a^2 + 1} + 1$$

$$\div a^4 \times \frac{a^4 - 4a^2}{6a^4 - 4a^2 + 1}; \text{ and if we put } \frac{a^4 - 4a^2}{6a^4 - 4a^2 + 1} = \pi, \text{ and}$$

the modulus of any kind of logarithms = M , the last expression will become $\frac{\pi a^4 + 1}{\pi a^4}$, and its logarithm will

$$\text{be } = \text{either } M \times : \frac{1}{\pi a^4 + 1} + \frac{1}{2 \cdot (\pi a^4 + 1)^2} + \frac{1}{3 \cdot (\pi a^4 + 1)^3} \&c.$$

$$\text{or } 2 M \times : \frac{1}{2\pi a^4 + 1} + \frac{1}{3 \cdot 2\pi a^4 + 1)^3} + \frac{1}{5 \cdot 2\pi a^4 + 1)^5} \&c. \text{ It}$$

therefore follows, that $4 \log. B + 4 \log. D - 6 \log. C - \log. A - \log. E$ is =

M

$$\left\{ M \times : \frac{1}{\pi a^4 + 1} + \frac{1}{2 \cdot \pi a^4 + 1} + \frac{1}{3 \cdot \pi a^4 + 1} + \dots \right\} \text{ which is THEOREM III.}$$

$$\left\{ 2M \times : \frac{1}{2\pi a^4 + 1} + \frac{1}{3 \cdot 2\pi a^4 + 1} + \frac{1}{5 \cdot 2\pi a^4 + 1} + \dots \right\} \text{ which is THEOREM IV.}$$

Before we make any observations on the several other expressions contained in the table of differences of logarithms, it may not be improper to ascertain the degrees of accuracy of the approximations to the values of the series given above.

As the second term of the series in theorem 2, is $\frac{2M}{3 \cdot 2pa^3 - 1} = \frac{2M}{24p^3a^9 - 36p^2a^6, \&c.}$ it is evident, that, if the first term, $\frac{2M}{2pa^3 - 1}$, be expressed in a series, all the terms of it in which $\frac{1}{a^9}$ is not found will be true, and that the error will take place in that term in which $\frac{1}{a^9}$ enters. Now, since the value of p , in series, is $\frac{1}{2} + \frac{3}{4a} - \frac{3}{8a^2} + \frac{3}{16a^3} - \frac{3}{32a^4} + \frac{3}{64a^5} \&c.$ $2pa^3 - 1$ will be $= a^3 + \frac{3}{2}a^2 - \frac{3}{4}a - \frac{5}{8} - \frac{3}{16}a^{-1} + \frac{3}{32}a^{-2} \&c.$ and $\frac{2M}{2pa^3 - 1}$ will be $= 2M \div : a^3 + \frac{3}{2}a^2 - \frac{3}{4}a - \frac{5}{8} - \frac{3}{16}a^{-1} + \frac{3}{32}a^{-2}, \&c. = M \times : \frac{2}{a^3} - \frac{3}{a^4} + \frac{6}{a^5} - \frac{10}{a^6}$

E

+

+ $\frac{18}{a^7} - \frac{63}{2a^8}$ &c. which fix terms, it is evident, are true. But $2M \div a^3 + \frac{3}{2}a^2 - \frac{3}{4}a - \frac{5}{8}$ is $= M \times : \frac{2}{a^3} - \frac{3}{a^4} + \frac{6}{a^5} - \frac{10}{a^6} + \frac{141}{8a^7}$ &c. which differs from the value of $2M \div \overline{2pa^3 - 1}$, expressed in series, by defect, the first term of the error being $-\frac{3M}{8a^7}$, which indeed, in the proper use of the theorem, will be a very small quantity.

In like manner it is discovered, that $2M \div \overline{\frac{1}{3}a^4 - \frac{10}{9}a^2 + \frac{11}{54}} = M \times : \frac{6}{a^4} + \frac{20}{a^6} + \frac{63}{a^8} + \frac{1780}{9a^{10}}$ &c. is less than $\frac{2M}{2\pi a^4 + 1}$, the first term of the series in theorem 4; and that the first term of the error is $-\frac{56M}{9a^{10}}$, which, when a is a large number, will be an exceedingly small quantity.

From what is done above it is obvious, that, as many of the leading terms as are taken in the divisors $a^3 + \frac{3}{2}a^2 - \frac{3}{4}a - \frac{5}{8}$ and $\frac{1}{3}a^4 - \frac{10}{9}a^2 + \frac{11}{54}$, so many terms will be obtained in the series $M \times : \frac{2}{a^3} - \frac{3}{a^4} + \frac{6}{a^5} - \frac{10}{a^6}$ &c. and $M \times : \frac{6}{a^4} + \frac{20}{a^6} + \frac{63}{a^8}$ &c. respectively.

And

And hence we have different approximations to the values of δ and Σ , the third and fourth differences of logarithms.

For the third Differences we have,

$$\begin{array}{l}
 1. \quad \frac{2M}{2pa^3 - 1} \\
 2. \quad \frac{4M}{2a^3 + 3a^2 - 1.5a - 1.25} \\
 3. \quad \frac{4M}{2a^3 + 3a^2 - 1.5a} \\
 4. \quad \frac{4M}{2a^3 + 3a^2} \\
 5. \quad \frac{2M}{a^3}
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{Which, when} \\ a \text{ is} = 1000, \\ \text{gives } \delta \text{ true to} \end{array}
 \left[\begin{array}{l} 27 \\ 21 \\ 18 \\ 15 \\ 12 \end{array} \right] \text{ Places of figures.}$$

For the fourth Differences we have,

$$\begin{array}{l}
 1. \quad \frac{2M}{2\pi a^4 + 1} \\
 2. \quad \frac{18M}{3a^4 - 10a^2 + 1.8\frac{1}{3}} \\
 3. \quad \frac{18M}{3a^4 - 10a^2} \\
 4. \quad \frac{6M}{a^4}
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{Which, when} \\ a \text{ is} = 1000, \\ \text{gives } \Sigma \text{ true to} \end{array}
 \left[\begin{array}{l} 35 \\ 29 \\ 23 \\ 17 \end{array} \right] \text{ Places of figures.}$$

If we look back to pages 22 and 23, we shall find, among the several algebraic expressions contained in

the column of *first Differences* of logarithms, these two, $\log. B - \log. C = \log. \frac{a+1}{a}$, and $\log. C - \log. D = \log. \frac{a}{a-1}$, from which the following well known series are derived :

$$\text{Log. } \frac{a+1}{a} = M \times : \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} - \frac{1}{4a^4} + \frac{1}{5a^5} \&c.$$

$$\text{Log. } \frac{a}{a-1} = M \times : \frac{1}{a} + \frac{1}{2a^2} + \frac{1}{3a^3} + \frac{1}{4a^4} + \frac{1}{5a^5} \&c.$$

And by taking the product of these numbers, and the sum of their logarithms, is obtained another series which converges twice as fast as either of the preceding ones : viz.

$$\text{Log. } \frac{a+1}{a-1} = M \times : \frac{2}{a} + \frac{2}{3a^3} + \frac{2}{5a^5} + \frac{2}{7a^7} \&c.$$

And by help of this series * the before mentioned two first differences of logarithms are expressed thus :

$$\text{Log. } \frac{a+1}{a} = 2 M \times : \frac{1}{2a+1} + \frac{1}{3 \times 2a+1)^3} + \frac{1}{5 \times 2a+1)^5} \&c.$$

$$\text{Log. } \frac{a}{a-1} = 2 M \times : \frac{1}{2a-1} + \frac{1}{3 \times 2a-1)^3} + \frac{1}{5 \times 2a-1)^5} \&c.$$

each of which series converges, in general, above twice as swiftly as either of those by which these logarithms were at first expressed.

* See *Simpson's Construction of Logarithms.*

In the column of *second differences* we find $2 \log C - \log. B - \log. D = \log. \frac{aa}{aa-1}$, which expression admits of an easy conversion into the following simple series, $M \times : \frac{1}{a^2} + \frac{1}{2a^4} + \frac{1}{3a^6} + \frac{1}{4a^8} \&c.$ But, it is evident, from what was done in page 28, that the same logarithm is expressed by this series, $2 M \times : \frac{1}{2aa-1} + \frac{1}{3 \times 2aa-1}^3 + \frac{1}{5 \times 2aa-1}^5 \&c.$ which, in general, will converge somewhat above twice as quick as that immediately preceding it.—And these, with one * for calculating a second difference from the sum of two first differences, are all the principal series for computing logarithms, that I know of, which have hitherto been published.

In the column of *third differences* we find but two expressions, from the first of which were derived the series given in theorems 1st and 2d in this Essay. The second expression is $3 \log. D + \log. B - 3 \log. C - \log. E = \frac{\overline{a-1}^3 \times \overline{a+1}}{a^3 \times a-2}$, from which the two following THEOREMS, (which may be numbered 7 and 8) are

* The series here meant is the quotient which arises by dividing the series expressing $\log. \frac{a+1}{a-1}$, by the series which expresses $\log. \frac{aa}{aa-1}$. It is one of Sir ISAAC NEWTON'S.

deduced

deduced by a method similar to that used in page 23.

For, putting $\frac{a-2}{2a-1} = q$, and the modulus $= M$, we obtain $3 \log. D + \log. B - 3 \log. C - \log. E =$ either

$$M \times : \frac{1}{qa^3+1} + \frac{1}{2 \times qa^3+1}^2 + \frac{1}{3 \times qa^3+1}^3 \text{ \&c. or } 2 M \times :$$

$$\frac{1}{2qa^3+1} + \frac{1}{3 \times 2qa^3+1}^3 + \frac{1}{5 \times 2qa^3+1}^5 \text{ \&c. And hence one}$$

might easily obtain five approximations similar to those on page 27.

In the column of *fourth differences* there is but one algebraic expression, from which were derived those very swiftly converging series which were given in theorem 3 and 4.

Having taken a view of the several expressions contained in the table, pages 22 and 23, and of the series, thence derived, for computing 1st, 2d, 3d, and 4th, differences of logarithms, independent of each other; let us now see what relation those differences have among themselves, as we shall, by that means, obtain several other theorems which will be useful in computing or examining a table of logarithms.

In order to this, it seems best to write down those differences in series of the simplest form they admit.

And retaining the notation which has been all-along used in this Essay, we have,

First

First Differences of Logarithms.

$$(\text{Log. } A - \text{log. } B =) M \times : \frac{1}{a} - \frac{3}{2a^2} + \frac{7}{3a^3} - \frac{15}{4a^4} + \frac{31}{5a^5} - \frac{63}{6a^6} \text{ \&c.}$$

$$(\text{Log. } B - \text{log. } C =) M \times : \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} - \frac{1}{4a^4} + \frac{1}{5a^5} - \frac{1}{6a^6} \text{ \&c.}$$

$$(\text{Log. } C - \text{log. } D =) M \times : \frac{1}{a} + \frac{1}{2a^2} + \frac{1}{3a^3} + \frac{1}{4a^4} + \frac{1}{5a^5} + \frac{1}{6a^6} \text{ \&c.}$$

$$(\text{Log. } D - \text{log. } E =) M \times : \frac{1}{a} + \frac{3}{2a^2} + \frac{7}{3a^3} + \frac{15}{4a^4} + \frac{31}{5a^5} + \frac{63}{6a^6} \text{ \&c.}$$

Second Differences of Logarithms.

$$M \times : \frac{1}{a^2} - \frac{2}{a^3} + \frac{14}{4a^4} - \frac{6}{a^5} + \frac{62}{6a^6} \text{ \&c.}$$

$$M \times : \frac{1}{a^2} \quad * \quad + \frac{2}{4a^4} \quad * \quad + \frac{2}{6a^6} \text{ \&c.}$$

$$M \times : \frac{1}{a^2} + \frac{2}{a^3} + \frac{14}{4a^4} + \frac{6}{a^5} + \frac{62}{6a^6} \text{ \&c.}$$

Third Differences of Logarithms.

$$M \times : \frac{2}{a^3} - \frac{3}{a^4} + \frac{6}{a^5} - \frac{10}{a^6} \text{ \&c.}$$

$$M \times : \frac{2}{a^3} + \frac{3}{a^4} + \frac{6}{a^5} + \frac{10}{a^6} \text{ \&c.}$$

Fourth Difference of Logarithms.

$$M \times : \frac{6}{a^4} \quad * \quad + \frac{20}{a^6} \text{ \&c.}$$

On inspecting this table of series, the following particulars are obvious.

1. That

1. That, of the four series expressing first differences of logarithms, the two middle ones, on account of the simplicity of their co-efficients, are more eligible for the purpose of numerical calculation, than the other two.

2. That, of these two middle series, that which has all its terms affirmative is more eligible to calculate by than the other, in which the terms are alternately affirmative and negative.

3. That, of the three series expressing second differences, the middle one is most eligible for calculation, on a double account: as the co-efficients of it are more simple than the co-efficients of either of the other two series, and as it converges twice as fast by the powers of a .

4. That, of the two series which express third differences, that in which the signs are all affirmative is the most eligible for numerical calculation. And,

5. That, of all the series in the table, none converges by a higher power of a than its square.

And if we compare those series in the above table which express third and fourth differences of logarithms, with the series in page 27 which express the same things, we shall see the agreement of the results of very different processes.

In

In making these observations one can hardly overlook the advantage which one series has over another, in point of convergency, though both express the same thing. The series in the above table, which express third differences, converge only by *single powers* of a ; the series given in theorems 1 and 2, converge, the one by *cubes* of a , the other by *sixth powers* of a . Again, the series in the table, which expresses fourth differences, converges only by *squares* of a ; the series given in theorems 3 and 4 converge, the one by *fourth powers* of a , the other by *eighth powers* of a . And it will be no difficult matter to resolve either of those very swiftly converging series into two others, of peculiar relation to each other, and not incumbered with co-efficients, which shall converge twice as fast; for instance, that which now converges by a^3 into two others which shall converge by $a^{1.6}$! But of this hereafter.

I come now to the principal use of the above table, To find the relation which the several differences of logarithms have to each other.

It is indeed equally easy to find what relation any one of the above series has to any other; but as all those relations would take up more room than is consistent with the plan of this work, I shall at present make no more use of the first and fourth of the first

F

differences,

differences, and the first and third of the second differences; the rest are denoted by the following characters :

$$\alpha = M \times : \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} - \frac{1}{4a^4} \&c.$$

$$\alpha' = M \times : \frac{1}{a} + \frac{1}{2a^2} + \frac{1}{3a^3} + \frac{1}{4a^4} \&c.$$

$$\beta = M \times : \frac{1}{a^2} + \frac{1}{2a^4} + \frac{1}{3a^6} + \frac{1}{4a^8} \&c.$$

$$\gamma = M \times : \frac{2}{a^3} - \frac{3}{a^4} + \frac{6}{a^5} - \frac{10}{a^6} \&c.$$

$$\gamma' = M \times : \frac{2}{a^3} + \frac{3}{a^4} + \frac{6}{a^5} + \frac{10}{a^6} \&c.$$

$$\delta = M \times : \frac{6}{a^4} + \frac{20}{a^6} + \frac{63}{a^8} \&c.$$

If now we put P for the quantity by which α being multiplied, the product shall be $=\beta$, we shall have $\alpha \times P = \beta$; and then, by division, $P = \beta \div \alpha, = M \times :$

$$\frac{1}{a^2} + \frac{1}{2a^4} + \frac{1}{3a^6} \&c. \div M \times : \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} \&c. =$$

$$\frac{1}{a} - \frac{1}{2a^2} + \frac{5}{12a^3} \&c. \text{ which value of } P \text{ being written}$$

for it in the former equation, we have a new one (the first equation below,) in which the relation of α to β is expressed by a series clear of M , which, in BRIGGS's logarithms, is a troublesome number. And in

like

like manner the other series in the following thirteen equations were obtained.

*For calculating SECOND DIFFERENCES from FIRST DIFFERENCES,
by a MULTIPLIER.*

$$1. \alpha \times : \frac{1}{a} + \frac{1}{2a^2} + \frac{5}{12a^3} + \frac{7}{24a^4} \&c. = \beta.$$

$$2. \alpha' \times : \frac{1}{a} - \frac{1}{2a^2} + \frac{5}{12a^3} - \frac{7}{24a^4} \&c. = \beta.$$

*For calculating THIRD DIFFERENCES from FIRST DIFFERENCES,
by a MULTIPLIER.*

$$1. 2\alpha \times : \frac{1}{a^2} - \frac{1}{a^3} + \frac{13}{6a^4} - \frac{10}{3a^5} \&c. = \gamma.$$

$$2. 2\alpha \times : \frac{1}{a^2} + \frac{2}{a^3} + \frac{11}{3a^4} + \frac{77}{12a^5} \&c. = \gamma'.$$

$$3. 2\alpha' \times : \frac{1}{a^2} - \frac{2}{a^3} + \frac{11}{3a^4} - \frac{77}{12a^5} \&c. = \gamma.$$

$$4. 2\alpha' \times : \frac{1}{a^2} + \frac{1}{a^3} + \frac{13}{6a^4} + \frac{10}{3a^5} \&c. = \gamma'.$$

*For calculating FOURTH DIFFERENCES from FIRST DIFFERENCES,
by a MULTIPLIER.*

$$1. 6\alpha \times : \frac{1}{a^3} + \frac{1}{2a^4} + \frac{13}{4a^5} \&c. = \delta.$$

$$2. 6\alpha' \times : \frac{1}{a^3} - \frac{1}{2a^4} + \frac{13}{4a^5} \&c. = \delta.$$

*For calculating THIRD DIFFERENCES from SECOND DIFFERENCES,
by a MULTIPLIER.*

$$1. 2\beta \times : \frac{1}{a} - \frac{3}{2a^2} + \frac{5}{2a^3} - \frac{17}{4a^4} \&c. = \gamma.$$

F 2

2.

$$2. 2\beta \times : \frac{1}{a} + \frac{3}{2a^2} + \frac{5}{2a^3} + \frac{17}{4a^4} \&c. = \gamma.$$

For calculating FOURTH DIFFERENCES from SECOND DIFFERENCES,
by a MULTIPLIER.

$$6\beta \times : \frac{1}{a^2} + \frac{17}{6a^4} + \frac{35}{4a^6} \&c. = \delta.$$

For calculating FOURTH DIFFERENCES from THIRD DIFFERENCES,
by a MULTIPLIER.

$$1. 3\gamma \times : \frac{1}{a} + \frac{3}{2a^2} + \frac{31}{12a^3} + \frac{35}{8a^4} \&c. = \delta.$$

$$2. 3\gamma' \times : \frac{1}{a} - \frac{3}{2a^2} + \frac{31}{12a^3} - \frac{35}{8a^4} \&c. = \delta.$$

But the relation which these several orders of differences have to each other may also be expressed by means of series that are to be used as *divisors*; which, in many cases, will be more commodious than those that are to be used as *multipliers*.

Of the several methods that might be used in finding such series, I take the following:

Put $\alpha \div \mathcal{Q} = \beta$; then will α be $= \beta \mathcal{Q}$, and $\mathcal{Q} = \alpha \div \beta$, $= M \times : \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} \&c. \div M \times : \frac{1}{a^2} + \frac{1}{2a^4} + \frac{1}{3a^6} \&c. = a - \frac{1}{2} - \frac{1}{6a} - \frac{1}{20a^3} \&c.$ And by a similar process the other series in the subsequent thirteen equations were found.

For

*For calculating SECOND DIFFERENCES from FIRST DIFFERENCES,
by a DIVISOR.*

$$1. \alpha \div : a - \frac{1}{2} - \frac{1}{6a} - \frac{1}{20a^3} \&c. = \beta.$$

$$2. \alpha' \div : a + \frac{1}{2} - \frac{1}{6a} - \frac{1}{20a^3} \&c. = \beta.$$

*For calculating THIRD DIFFERENCES from FIRST DIFFERENCES,
by a DIVISOR.*

$$1. 2\alpha \div : a^2 + a - \frac{7}{6} \pm * \&c. = \gamma.$$

$$2. 2\alpha \div : a^2 - 2a + \frac{1}{3} + \frac{1}{4a} \&c. = \gamma'.$$

$$3. 2\alpha' \div : a^2 + 2a + \frac{1}{3} - \frac{1}{4a} \&c. = \gamma.$$

$$4. 2\alpha' \div : a^2 - a - \frac{7}{6} \pm * \&c. = \gamma'.$$

*For calculating FOURTH DIFFERENCES from FIRST DIFFERENCES,
by a DIVISOR.*

$$1. 6\alpha \div : a^3 - \frac{1}{2}a^2 - 3a + \frac{17}{12} \&c. = \delta.$$

$$2. 6\alpha' \div : a^3 + \frac{1}{2}a^2 - 3a - \frac{17}{12} \&c. = \delta.$$

*For calculating THIRD DIFFERENCES from SECOND DIFFERENCES,
by a DIVISOR.*

$$1. 2\beta \div : a + \frac{3}{2} - \frac{1}{4a} + \frac{1}{8a^2} \&c. = \gamma.$$

$$2. 2\beta \div : a - \frac{3}{2} - \frac{1}{4a} - \frac{1}{8a^2} \&c. = \gamma'.$$

For

For calculating FOURTH DIFFERENCES *from* SECOND DIFFERENCES,
by a DIVISOR.

$$6\beta \div : a^2 - \frac{17}{6} - \frac{13}{18a^2} \&c. = \delta.$$

For calculating FOURTH DIFFERENCES *from* THIRD DIFFERENCES,
by a DIVISOR.

$$1. \quad 3\gamma \div : a - \frac{3}{2} - \frac{1}{3a} \pm * \&c. = \delta.$$

$$2. \quad 3\gamma' \div : a + \frac{3}{2} - \frac{1}{3a} \pm * \&c. = \delta.$$

The use of the above twenty-six equations, as well as the greater facility of calculating by the latter thirteen of them than by the former, is too obvious to an Algebraist to need any explanation.

Let us now come to the CONSTRUCTION *of a* TABLE
of BRIGGS'S LOGARITHMS.

Here the first thing to be done is to find the modulus, which may be performed several ways. An Algebraist might easily do it by help of the theorems given in this Essay. Or, the hyperbolic logarithm of 10, which is well known to be the reciprocal of the modulus, might be quickly computed by some of the theorems in Essay I. Or, a calculator who would
avail

avail himself of his former work, might very easily compute the hyp. log. of 10 thus: Since *b. l.* 3 was computed in page 7, the *b. l.* 9 is also known; and if to that we add *b. l.* $\frac{10}{9}$, which is $-0.1 + \frac{0.01}{2} + \frac{0.001}{3} + \frac{0.0001}{4} + \frac{0.00001}{5}$ &c. the *b. l.* 10 will be had. Or, it may yet be computed in a different manner by means of the 1st, 3d, 5th, &c. powers of $\frac{1}{9}$; which, as it is new to me, I will here give.

Put $\frac{1}{9}, \frac{1}{9^3}, \frac{1}{9^5}, \&c. = A, B, C, \&c.$ then will $\frac{1}{2}$ *b. l.* $\frac{5}{4}$ be $= A + \frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D, \&c.$ which is a well known series; and $\frac{1}{2}$ *b. l.* 2 will be =

$$\left\{ \begin{array}{l} + 13\frac{1}{3} \times : \frac{A}{1.3} + \frac{2B}{5.7} + \frac{3C}{9.11} + \frac{4D}{13.15} + \frac{5E}{17.19} \&c. \\ - 4 \times : \frac{A}{1.3} + \frac{B}{5.7} + \frac{C}{9.11} + \frac{D}{13.15} + \frac{E}{17.19} \&c. \end{array} \right.$$

of which series the investigation shall be given hereafter. The numerical work may stand thus:

For the Hyperbolic Logarithm of $\frac{5}{4}$.

$$\begin{array}{l} A = 0.1111111111111111 \\ \frac{1}{3} B = 0.000457247370828 \\ \frac{1}{5} C = 0.000003387017562 \\ \frac{1}{7} D = 0.000000029867880 \\ \frac{1}{9} E = 0.000000000286797 \\ \frac{1}{11} F = 0.000000000002897 \\ \frac{1}{13} G = 0.000000000000030 \end{array}$$

The sum is $\frac{1}{2}$ *b. l.* $\frac{5}{4} = 0.111571775657105 = m.$

For

For the Hyperbolic Logarithm of 2.

$\frac{A}{1.3} = 0.037037037037037$	$\frac{A}{1.3} = 0.037037037037037$
$\frac{B}{5.7} = 0.000039192631785$	$\frac{2B}{5.7} = 0.000078385263570$
$\frac{C}{9.11} = 0.000000171061493$	$\frac{3C}{9.11} = 0.000000513184479$
$\frac{D}{13.15} = 0.000000001072180$	$\frac{4D}{13.15} = 0.000000004288721$
$\frac{E}{17.19} = 0.000000000007991$	$\frac{5E}{17.19} = 0.000000000039956$
$\frac{F}{21.23} = 0.00000000000066$	$\frac{6F}{21.23} = 0.00000000000396$
Sum $\times$ } 0.037076401810552	$\frac{7G}{25.27} = 0.00000000000004$
- 4 is } -4	Sum $\times$ { 0.037115939814163
$= -0.148305607242208$	$\frac{13 \frac{1}{3}}{10 + \frac{1}{3}} = \left\{ \begin{array}{l} 0.012371979938054 \\ 0.012371979938054 \end{array} \right.$
$+ 0.49487919752217$	is $= + 0.49487919752217$
$\frac{1}{2} b.l. 2 = 0.34657359027996 = n$	

And then, $2.07944154167976 = 6n$
 $0.22314355131421 = 2m;$

And $H. L. 10 = 2.30258509299397 (= 2m + 6n)$, from which we get $M = 0.43429448190326$. And this number is sufficiently exact for the purpose of computing a Table of BRIGGS's logarithms to fourteen places besides the index, which is the limit I propose for the logarithms in the following specimen.

The next thing to be determined, is the extent of the table, or for what and how many numbers the logarithms are to be computed. The limits I shall set are 10000 and 100000, since it is evident that, if the logarithms of these two numbers, and of all the intermediate ones differing from each other by 1, be once obtained, the logarithms of all the whole numbers below 10000 will be had without any further labour than taking 1, 2, 3, or 4, from the index, as the case shall require.

Briggs's logarithm of 100000 is known to be 5; and and if we put A, B, C, D , and $E = 100000, 99999, 99998, 99997$, and 99996 respectively, and denote the first of the 1st, 2d, 3d, and 4th orders of differences by d, e, f , and g , respectively, we shall have

$$d = 0.00000434296653'3902,$$

$$e = 0.00000000004343'031679,$$

$$f = 0.00000000000000'0868628,$$

$g = 0.00000000000000'000002606$, nearly; which values are easily computed by means of what was delivered in the former part of this Essay.

Now, at first view, it seems as if a great many logarithms might be computed by only adding and subtracting these differences. A considerable number of them indeed may be so computed; but we must proceed with caution. It is well known, that,

G

if

if the logarithm of A be $= L$, then shall the logarithm of $A - n = L - nd - n \cdot \frac{n-1}{2} e - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} f - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} g$, to a limited number of places of figures; where it is evident that the number of true figures will be the less as n is greater. To illustrate this matter, I have added the following table, in which the co-efficients of d , e , f , and g are exhibited in the 1st, 2d, 3d and 4th columns, respectively, when $n = 10, 20, 30, \&c.$

n	$n \cdot \frac{n-1}{2}$	$n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$	$n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$
10	45	120	210
20	190	1140	4845
30	435	4060	27405
40	780	9880	91390
50	1225	19600	230300
&c. to			
100	4950	161700	3921225

Now, by inspecting this table it appears, that, when n is $= 100$, the co-efficient of g is no less than 3921225, which is the number of times that the fourth difference must be subtracted in that case; and consequently, g must be calculated to at least seven places more than the logarithms are to have, that they may be true in their last figures. But if n be taken $= 50$, then the
co-efficient

co-efficient of g will be $= 230300$, and $230300 g$, taking for g the value above written, will give a 6 in the fifteenth place of decimals. It appears also, that 50, 1225, and 19600, the other numbers in the same line with 230300, are the co-efficients of d , e , and f , respectively, when $n = 50$, and serve to point out to how many more places of figures each of those differences ought to be calculated, than the logarithms are to have.

These precautions being taken, fifty logarithms may be computed very expeditiously, as follows: add the 4th difference continually to the 3d, every 3d difference to the preceding 2d difference, and every 2d difference to the preceding 1st difference, and subtract every 1st difference from the preceding logarithm; continue thus till fifty remainders are obtained, which will be the logarithms of so many numbers, viz. from 100000 to 99950 inclusive. A specimen of this arithmetical work is given in the following table.

COMPUTATION OF LOGARITHMS.

Numbers.	Logarithms.	1st Differences.	2d Differences.	3d Differen.	4th D.
100000	5.00000000000000000000	434296653 '3902	4343 '031679	'0868628	26
99999	4.99999565703346 '6098	434300996 '4219	4343 '118542	'0868654	26
99998	4.99999131402350 '1879	434305339 '5404	4343 '205407	'0868680	26
99997	4.99998697097010 '6475	434309682 '7458	4343 '292275	'0868706	26
99996	4.99998262787327 '9017	434314026 '0381	4343 '379146	'0868732	26
99995	4.99997828473301 '8636	434318369 '4172	4343 '466019	'0868758	26
99994	4.99997394154932 '4464	434322712 '8832	4343 '552895	'0868784	26
99993	4.99996959832219 '5632	434327056 '4361	4343 '639773	'0868810	26
99992	4.99996525505163 '1271	434331400 '0759	4343 '726654	&c.	&c.
99991	4.99996091173763 '0512	434335743 '8026	&c.		
99990	4.99995656838019 '2486				
&c. to	&c.				
99950	4.99978279845413 '573				

2d differences true to the nearest unit in the last figure. And some other abridgements may be used in these computations, which need not be pointed out to able calculators.

Note 1. In this table, the inverted comma between the fourteenth and fifteenth places of decimals serves to point out the proper position of the numbers for addition and subtraction. The two figures in the 4th difference, being removed from the comma to an equal distance with the last two figures of the 3d difference, the comma and ciphers are omitted, to save room.

Note 2. It was not necessary to retain so many places of figures beyond the 14th, in the 2d differences, as in the 3d; nor so many in the 1st differences as in the 2d; but only to set down the 1st and

Note 3. These logarithms are true to sixteen places of figures, the advantage of which will appear when we come a little further on: but to obtain them to fourteen places only, it will be sufficient to retain two places of figures less in each of the decimal columns, except the 4th difference, provided that the last figure be always set down to the nearest unit.

A hundred more logarithms may also be computed by the addition and subtraction of differences, as soon as the logarithm of 99900, and the differences are calculated. But the $\log. 99900 = \log. 100000 - \log. \frac{10000}{999}$; and $\log. \frac{10000}{999}$ is found by a very easy calculation to be $= 0.00043451177401769$. The 1st, 2d, 3d, and 4th differences of the logarithms of the five arithmetic progressions, whose mean is 99900 and common difference 1, are also easily calculated by the series and approximations given in the former part of this essay. These requisites, properly placed, will stand as below.

Numbers.	Logarithms.	1st Differences.	2d Differences.	3d Differ.	4th D.
99950 &c. to	4.99978279845413.575				
99901					
99900	4.99956548822598.2309	434727035.30701	4351.643755	0871187	26.16
99899					
&c. to 99850					

Note. In this table, 26, the first two figures of the 4th difference, are removed from the point to an equal distance with 87, the last two figures of the 3d difference; and the 16, cut off from them by an in-

verted comma at the bottom of the line, will make about 1 to be added every sixth time; i. e. 27 must then be written instead of 26. The other points mark the same things here as in the preceding table.

Now, the logarithms of the numbers from 99900 up to 99950, are to be computed by subtracting the 4th differences successively from the 3d, the 3d differences from the 2d, and these again from the 1st; and by adding each 1st difference to the preceding logarithm.

Here it is obvious that the logarithm of 99950 will be obtained by two different calculations, which will serve to verify the work, or discover whether any error has crept into it. But indeed any intermediate logarithm, or difference, may be readily proved by means of the well-known binomial theorem; and if n be taken = 10, 20, &c. to 50, the co-efficients may be found ready calculated in page 42.

The calculation of the fifty logarithms of the numbers from 99900 down to 99850 inclusive, will be altogether similar to that explained above for fifty logarithms of the numbers from 100000 down to 99950.

Another 100 logarithms, of the numbers 99850, 99849, &c. down to 99750, may be obtained in like manner, the logarithm of 99800, (which is = $\log. 100000 - \log. \frac{1000}{998}$), and one of the several orders of differences to the 4th, being first computed. Another 100 may be obtained from the logarithm of 99700, and the proper differences: and so on, as far as is requisite.

But, although the calculations of the logarithms of 99800, 99700, &c. are very easily made by means of
of

of the fractions $\frac{1000}{998}$, $\frac{1000}{997}$, &c. yet these logarithms may be more easily computed by making use of the second and fourth theorems in this essay, or of the approximations in page 27.

This will appear in the following example, where, one difference of the 1st, 2d, and 3d orders, and seven of the 4th order being calculated, ten logarithms are had by addition and subtraction only. These differences are exhibited in the following table.

COMPUTATION OF LOGARITHMS.

Num.	Logarithms.	1st Differences.	2d Differences.	3d Differences.	4th Differences.
1000	3.0000000000000000	4345117401	43516459	87251	262
999		'7690	'3508	'06822	'672647
998					263
997					'728087
996					264
995					'788834
994					265
993					'854919
992					266
991					'926375
990					268
					'003235
					269
					'085530

Note. The inverted comma at the top of the lines in this table, serves to point out the position of the numbers for addition and subtraction, as has been mentioned in the notes to the preceding tables.

We have, therefore, only to add and subtract the differences given in the last table, in the manner already described, to obtain the logarithms of 999, 998, &c. down to 990. And by adding 2 to the index of these logarithms, those of the ten arithmetic progressionals 99900, 99800, &c. to 99000 are obtained.

The logarithms of 100000 and 99900 were interpolated above, by inserting 99 others between them; and when the rest are interpolated in the same manner, and the several orders of differences applied downward for fifty numbers below 99000, there will be had 1050 logarithms; viz. of all the numbers from 100000 to 98950.

And in the same manner that the logarithms of the ten numbers from 1000 to 990 were computed, one might compute those of 989, 988, &c. to 980; and then, by interpolation, get another thousand or logarithms: and so on.

But it will presently appear, that the logarithms of 989, 988, &c. to 980 may be much more easily obtained from the 1050 logarithms already computed. For, as the logarithms of 99990 and 990 have been computed, and $99990 \div 990 = 101$, the logarithm of 101 is given: and as the logarithms of these ten progressional numbers 99889, 99788, 99687, &c. to 98980, are among those which were calculated, and

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these

these arithmetic progressions divided, separately, by 101 give 989, 988, 987, &c. to 980, the logarithms of this new set of progression numbers will be immediately had by taking the logarithm of 101 from those of the former set. And then, by completing the interpolation between 99000 and 98900, and interposing 99 intermediate logarithms between those of every two of the progressions 98900, 98800, &c. to 98000, and continuing the operation for fifty numbers below 98000, there will be obtained 2050 logarithms.

Now, since the logarithms of all the numbers from 98980 to 97950 are obtained, we need only to subtract the logarithm of $\frac{101}{100}$ from those of the ten progressions 98879, 98778, &c. to 97970, to obtain the logarithms of these ten 97900, 97800, &c. to 97000; and then by interpolation, as described above, there will be had in all 3050 logarithms.

And in this easy manner may we derive the logarithms of all the rest of the progressions differing from each other by 100, from 97000 to 50000, and compute the intermediate ones by the method of interpolation explained and illustrated above. The logarithms of these progressions indeed, cannot be had always by ten at a time, but by ten, nine, eight, seven, six, and five at a time, in succession. For as the

the numbers 99990 and 99000 are constantly diminished in the ratio of 101 to 100, their difference 990, as well as that difference increased by 50, i. e. 1040, will be lessened by 1 every time that 101 is taken from the former number and 100 from the latter. This appears very plainly in the following table, in which the numbers 99990 and 99000, and the remainders after the several equimultiples of 101 and 100 have respectively been taken from them, as well as the differences of the remainders after each subtraction, are exhibited.

<i>Equimultiples.</i>		
<i>of 101.</i>	<i>Diff.</i>	<i>of 100.</i>
99990	990	99000
$4 \times 1010 = 4040$		$4000 = 4 \times 1000$
95950	950	95000
$11 \times 909 = 9999$		$9900 = 11 \times 900$
85951	851	85100
$12 \times 808 = 9696$		$9600 = 12 \times 800$
76255	755	75500
$15 \times 707 = 10605$		$10500 = 15 \times 700$
65650	650	65000
$16 \times 606 = 9696$		$9600 = 16 \times 600$
55954	554	55400
$10 \times 505 = 5050$		$5000 = 10 \times 500$
50904	504	50400
404		400
50500	500	50000

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This

This table needs little or no explanation. It may, however, be remarked, that 1010 and 1000, as well as 909 and 900, &c. show how much the numbers in the same column with them are diminished every time that a set of ten, nine, &c. arithmetic progressionals is taken. It may be further observed, that the numbers 4, 11, 12, &c. standing before 1000, 900, 800, &c. serve to show how many sets of arithmetic progressionals of ten, nine, eight, &c. together, are taken; and that the sum of all the numbers 4, 11, 12, 15, 16, and 10, which is 68, shows the number of deductions of logarithms from one another by periods of, first, a thousand each, then, nine hundred, eight hundred, &c. till we arrive at the logarithm of 50400; and then one deduction more, of the logarithms of only four arithmetic progressionals, will supply us with all that are wanted for interpolation till we arrive at the logarithm of 50000.

And the knowledge of the number of times the logarithms were deduced from each other by periods of 1000, 900, 800, &c. will enable us to ascertain the degree of accuracy to be expected in them. For, as the accuracy of each of those periods of logarithms depended upon that of the preceding one, and the first thousand, as well as all the rest, were calculated to sixteen places of figures; and as there have been
but

but 69 deductions, by subtraction, to arrive at the logarithm of 50000, it is evident that these logarithms ought to be true to at least fourteen places of figures, besides the index.

And having got thus far, the remaining 40000 logarithms, viz. of the numbers from 50000 to 10000, may be had quickly by means of the logarithm of 2, which is now given. For the progression numbers 99998, 99996, &c. to 50000, being divided by 2, give 49999, 49998, &c. to 25000. Again, 49998, 49996, &c. to 25000, divided by 2, give 24999, 24998, &c. to 12500; and 24998, 24996, &c. to 20000, divided by 2 give 12499, 12498, &c. to 10000, the end of the table.

Of proving the Work, and checking Errors.

With regard to verifying the work and checking errors, that has been in a great measure provided for already, by the coincidence of the two logarithms, obtained by different calculations, of every 100th number from 99950 to 50050 inclusive. However, various other methods may quickly be devised for proving the work and checking errors, some of the most obvious of which here follow.

Since the logarithm of $\frac{100}{99}$ is known, and the arithmetic progressionals 100000, 99900, &c. divided

vided by $\frac{100}{99}$ give 99000, 98901, &c. we have only to take the logarithm of $\frac{100}{99}$ from those of the former numbers to obtain the logarithms of the latter, which will be checks on every 99th logarithm computed by the method above described. And several other checks of this kind might be used.

The logarithms of all the whole numbers from 100 to 50 would be so many other useful checks. These may be had from the first 2000 logarithms, and would be checks to every 10th logarithm of these arithmetic progressionals, 1000, 999, 998, &c. to 500; the logarithms of which progressionals, on account of their use, and for the sake of brevity, I shall in future call *fundamental logarithms*.

Now since $\frac{100000}{98000} = \frac{50}{49}$, it is evident to any one acquainted with the doctrine of ratios, that the fractions $\frac{100}{99}, \frac{99}{98}, \frac{98}{97}$, &c. to $\frac{51}{50}$, may be expressed by others whose numerators and denominators also shall be amongst the arithmetic progressionals differing from each other by 1, of which the first is 100000 and last 98000. Thus, $\frac{100}{99} = \frac{100000}{99000}, \frac{99}{98} = \frac{99990}{98980}, \frac{98}{97} = \frac{99960}{98940}$, &c. to $\frac{51}{50} = \frac{99960}{98000}$. Consequently, their logarithms may be had immediately from the first two thousand. And then, because $100 \div \frac{100}{99} = 99, 99 \div \frac{99}{98} = 98$, &c. the logarithms

rithms of 99, 98, &c. will be obtained one after another by the easy operation of subtraction.

But the logarithms of the numbers from 100 to 50 may be computed in a different and easier manner, by first calculating the logarithms of 2, 3, and some other small prime numbers. These also may be obtained several ways from the first 2000 logarithms, or even from the first 1050 only. I will point out the method of computing a few of them, by which the computations of the rest may be easily understood.

I begin with calculating the logarithm of 2 by means of some to be found amongst the first 1050 logarithms.

From the logarithms of 9898 and 101, computed above, we have the logarithm of 98; for $9898 \div 101 = 98$. And as $9996 \div 98 = 102$, the logarithm of 102 will be given. Then, because $2^{10} = 1024$, we shall have this fraction, $\frac{1024}{1020} = \frac{256}{255}$, whose logarithm may be had from two amongst the first 550 only, thus: $100000 \div 256$ gives a quotient of 390, and a remainder of 160; and 256×390 and 255×390 produce 99840 and 99450, which two numbers are amongst the 550 at the head of the table, and are the numerator and denominator of a new fraction $= \frac{1024}{1020}$. So that, if to the difference of the logarithms of 99840 and

99450

99450 the logarithm of 1020 be added, the sum will be the logarithm of 1024, $\frac{1}{10}$ of which is the logarithm of 2.

The logarithm of 2 being now found, those of 51 and 7 are quickly had by means of it; for $\frac{102}{2} = 51$, and $\sqrt[98]{2} = 7$.

The logarithm of 3 may be obtained from two logarithms amongst the second thousand, by means of those of 2 and 7, thus: $714 = 2401$, and $\frac{2401}{2400} (= \frac{2401}{3 \times 800}) = \frac{98441}{98400}$. It may be derived otherwise from two logarithms in the first two thousand, by means of the fraction $\frac{81}{80}$, which is $= \frac{99954}{98720}$. Or it may be had from four logarithms in the first thousand only, by means of the fractions $\frac{162}{161}$ & $\frac{161}{160}$, which are $= \frac{99954}{99337}$ & $\frac{99981}{99360}$ respectively. And this method may be used for obtaining the logarithm of any prime between 100 and 50.

The logarithm of 3 being found, those of 11 and 17 are immediately obtained, as $\frac{99}{9} = 11$, and $\frac{51}{3} = 17$. And the logarithms of other primes may be derived in like manner; as the logarithm of 41 from those of 7 and $98441, \frac{98441}{2401}$ being $= 41$; which method of deriving one logarithm from another, by means of the quotients

tients and products of numbers, is obvious and well known. Thus much for checks.

Easier Methods of computing the greater part of the Table.

But, although the method of computing a table of logarithms, above described, be an easy one, there may yet be found means of shortening the labour very considerably. These are, the derivation of all the fundamental logarithms immediately from the first 2000, and another method of interpolation.

The method of deriving the fundamental logarithms from the first 2000 only, will clearly appear in the following tables.

$$\left. \begin{array}{r} 980 \\ 979 \\ 978 \\ 977 \\ 976 \\ 975 \\ 974 \\ 973 \\ 972 \\ 971 \\ 970 \end{array} \right\} \times 102 = \left\{ \begin{array}{r} 99960 \\ 99858 \\ 99756 \\ 99654 \\ 99552 \\ 99450 \\ 99348 \\ 99246 \\ 99144 \\ 99042 \\ 98940 \end{array} \right.$$

$$970 \times 103 = 99910$$

In this table it appears, not only that ten fundamental logarithms are directly derived from an equal number amongst the first 2000, by means of the logarithm

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rithm of 102, but that the logarithm of 103 also is derived from another amongst them, by means of the logarithm of 970.

And in the following table, in which only the first and last progression number that is to be used with the same factor is set down, it appears, that all the rest of these logarithms are obtainable in the same easy manner.

$$\begin{array}{l} 970 \\ 960 \end{array} \} \times 103 = \left\{ \begin{array}{l} 99910 \\ 98880 \end{array} \right.$$

$$\begin{array}{l} 960 \\ 950 \\ \&c. \end{array} \} \times 104 = \left\{ \begin{array}{l} 99840 \\ 98800 \\ \&c. \end{array} \right.$$

$$\begin{array}{l} 520 \\ 518 \end{array} \} \times 190 = \left\{ \begin{array}{l} 98800 \\ 98420 \end{array} \right.$$

$$\begin{array}{l} 518 \\ 510 \end{array} \} \times 193 = \left\{ \begin{array}{l} 99974 \\ 98430 \end{array} \right.$$

$$\begin{array}{l} 510 \\ 500 \end{array} \} \times 196 = \left\{ \begin{array}{l} 99960 \\ 98000 \end{array} \right.$$

In the above tables it appears, that the logarithms of the factors 102, 103, 104, &c. are obtained by this method as fast as they are wanted: (many of them, however, may be more easily had by means of those of the small numbers 2, 3, 5, &c.) It is also evident, that, as the greatest factor is but 196, the fundamental

mental logarithms deduced in this manner will be true to fourteen places at least, besides the index, provided that the first 2000 logarithms were true to sixteen besides the index. The advantage of deriving the fundamental logarithms by this method consists in this: That, being all deduced immediately from the first 2000 logarithms, and not successively from one another, like those in page 50, it will be sufficient to insert the intermediate ones true to only fourteen places of decimals; by which a great many figures may be saved in the work of interpolation.

And with regard to checks, the same that were described above may be used here.

I come now to explain the other method of interpolation, which indeed is the same, in effect, as Sir *Isaac Newton's*.

It is well known, that

$$\text{Log. } \frac{a+1}{a} = M \times : \frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} - \frac{1}{4a^4} + \frac{1}{5a^5}, \&c.$$

$$\text{Log. } \frac{a}{a-1} = M \times : \frac{1}{a} + \frac{1}{2a^2} + \frac{1}{3a^3} + \frac{1}{4a^4} + \frac{1}{5a^5}, \&c.$$

and that

$$\text{Log. } \frac{a+x}{a} = M \times : \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5}, \&c.$$

$$\text{Log. } \frac{a}{a-x} = M \times : \frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{3a^3} + \frac{x^4}{4a^4} + \frac{x^5}{5a^5}, \&c.$$

which series differ from the former only by having

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the powers of x instead of those of 1; and which, when x is less than 1, will have the swifter convergency.

Now, since the first five terms of the series are sufficient for our purpose, if $P, Q, R, S,$ and T be written for $\frac{M}{a}, \frac{M}{2a^2}, \frac{M}{3a^3}, \frac{M}{4a^4},$ & $\frac{M}{5a^5},$ respectively, these logarithms will be expressed thus:

$$\text{Log. } \frac{a+1}{a} = P - Q + R - S + T,$$

$$\text{Log. } \frac{a}{a-1} = P + Q + R + S + T;$$

and

$$\text{Log. } \frac{a+x}{a} = Px - Qx^2 + Rx^3 - Sx^4 + Tx^5,$$

$$\text{Log. } \frac{a}{a-x} = Px + Qx^2 + Rx^3 + Sx^4 + Tx^5.$$

Here it is evident, that, if any easy method can be devised of finding the values of $P, Q,$ &c. in the former series, the logarithms expressed by the latter will be easily obtained, by only writing for x its proper value.

Now, by putting $\alpha, \alpha',$ &c. to signify the same things here as in page 34, and $\varepsilon =$ the 5th difference of logarithms, we shall have these equations:

1st Differences of Loga.	2d Difference.	3d Differences.	4th Difference.	5th Difference.
$P - \mathcal{Q} + R - S + T = \alpha$	$2\mathcal{Q} * + 2S * = \beta$	$6R - 12S + 30T = \gamma$	$24S * = \delta$	$120T = \epsilon$
$P + \mathcal{Q} + R + S + T = \alpha'$		$6R + 12S + 30T = \gamma'$		$120T = \epsilon$

And by taking the sums of the two first, and two third differences, we shall have the following five

equations for determining the values of $P, \mathcal{Q}, R, \&c.$ those of $\alpha, \alpha', \beta, \&c.$ being given.

$$2P + 2R + 2T = \alpha + \alpha'.$$

$$2\mathcal{Q} + 2S = \beta.$$

$$12R + 60T = \gamma + \gamma'.$$

$$24S = \delta.$$

$$120T = \epsilon.$$

And from these equations, (α' being $= \alpha + \beta$, and $\gamma' = \gamma + \delta$) we most easily get

$$\begin{aligned} P &= \alpha + \frac{1}{2}\beta - R - T = \alpha + \frac{1}{2}\beta - \frac{1}{6}\gamma + \frac{1}{12}\delta + \frac{1}{30}\epsilon. \\ \mathcal{Q} &= \frac{1}{2}\beta - S \dots = \frac{1}{2}\beta - \frac{1}{24}\delta. \\ R &= \frac{1}{6}\gamma + \frac{1}{12}\delta - 5T = \frac{1}{6}\gamma + \frac{1}{12}\delta - \frac{1}{24}\epsilon. \\ S &= \frac{1}{24}\delta. \\ T &= \frac{1}{120}\epsilon. \end{aligned}$$

And it is evident, that the values of $P, \mathcal{Q}, R, \&c.$ may be yet otherwise expressed, whenever occasion shall require it.

And for the same reason that the 1st, 2d, 3d, 4th, and 5th differences of the logarithms of the seven arithmetic progressionals whose mean is a , and common difference is α , will be

1st Differences of Logarithms.	2d Difference.	3d Differences.	4th Difference.	5th Differences.
$Px - 2x^2 + Rx^3 - Sx^4 + Tx^5$	$2Qx^2 * + 2Sx^4 *$	$6Rx^3 - 12Sx^4 + 30Tx^5$		$120Tx^5$
$Px + 2x^2 + Rx^3 + Sx^4 + Tx^5$		$6Rx^3 + 12Sx^4 + 30Tx^5$	$24Sx^4 *$	$120Tx^5$

Now, as the labour of calculating these differences will depend much upon the value of x , (and that value is in a great measure arbitrary,) it is manifest that the calculations will be shortest and easiest when x is $= \frac{1}{10}, \frac{1}{100}, \&c.$

EXAMPLE. Let x be $= \frac{1}{100}$, and let us take the differences, as far as the 5th order, of the logarithms of the numbers in arithmetic progression, whose mean is 1000, and common difference 1; then will

$$\begin{aligned} a' &= 0.00043451177401,769 \\ \beta &= 0.00000043429469,905 \\ \gamma' &= 0.00000000086989,446 \\ \delta &= 0.0000000000260,578 \\ \epsilon &= 0.0000000000001,045 \end{aligned}$$

From which we get

$$\begin{aligned} P &= 0.00043429448190,324 \\ Q &= 0.00000021714724,095 \\ R &= 0.00000000014476,483 \\ S &= 0.00000000000010,857 \end{aligned}$$

And then

$$\begin{aligned} \frac{1}{100} P &= 0.0000434294481,90324 \\ \frac{1}{10000} Q &= 0.0000000002171,47241 \\ \frac{1}{1000000} R &= 0.000000000000,01447 \end{aligned}$$

$$0.00000434296653,39012 = \text{Log.} \frac{100000}{99999}$$

This is the first difference in the following table. The other differences are such as are there set down, and will be found to agree with those in page 44, due regard being had to the different places in which the 3d and 4th differences stand.

1st Differences of Logarithms.	2d Differences.	3d Differ.	4th Diff.
0.00000434296653,39012	4342,944819	,0868602	26

These differences, it is obvious, are to be used, for fifty logarithms, in the manner described in page 43; and the logarithms obtained by them will be true to sixteen places besides the index, which is the degree of accuracy requisite for our purpose in the first 2000 logarithms*.

The interpolation of the logarithms of 1000 and 999, or indeed of any two fundamental logarithms, by interposing 99 others between them, may be performed another way, by inserting, first, nine logarithms between them, and then again nine between every two; in both which cases x will be $= \frac{1}{10}$.

This method, at first view, may seem to be easier than the other; for, as each set of differences is to be used a smaller number of times, fewer figures beyond the inverted comma will be requisite in each column. But when the additional work of calculating

* It is obvious that, in general, the differences will be most advantageously employed when they are used to find an equal number of logarithms both above and below that which stands on the same line with the 2d and 4th difference. In this place, the differences being those at the end of the table, there was no need to apply them upwards:

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the several different values of P , Q , &c. or at least of calculating the several sets of differences, is put in opposition to that saving, the advantage of this method will disappear.

But here the reader will perhaps be agreeably surprised to find, that the labour of inserting the first nine logarithms between every two of the fundamental ones, by the method just now described, may be avoided from the logarithm of 909 to that of 500, as the logarithm of every tenth number from 90900 to 50000 may be found much more easily. This will be made evident by the following table.

99990	} divided by 11, gives {	9090
99979		9089
99968		9088
99957		9087
99946		9086
99935		9085
99924		9084
99913		9083
99902		9082
99891		9081
&c. to		&c. to
55000		5000

The logarithms of the numbers in the right hand column might indeed be otherwise derived by a method similar to that used in page 57; but, the fundamental

damental logarithms being first calculated to 14 places of figures besides the index, and the intermediate ones, as far as they have been inserted, having the same degree of accuracy, the above method of obtaining the first 9 intermediate logarithms between every two of the fundamental ones from 909 to 500, is easier in practice, and exact enough*.

Now the insertion of 9 logarithms between every two of those belonging to the numbers in the right-hand column of the above table, by the second method of interpolation, will be easy work. For $\frac{1}{10} \delta$ having no value in 15 places of decimals, it is to be considered as = 0, and omitted, by which the calculations will be facilitated.

But before I give an example of this interpolation, it will not be improper to set down some yet more convenient *formulæ* for calculating the several sets of differences requisite in it.

The value of x being $\frac{1}{10}$, it is evident that the several orders of differences for our present purpose will become

1st Differences of Logarithms.	2d Difference.	3d Differences
$\frac{1}{10} P - \frac{1}{100} Q + \frac{1}{1000} R$	$\frac{2}{100} Q$	$\frac{6}{1000} R$
$\frac{4}{10} P + \frac{1}{100} Q + \frac{1}{1000} R$		$\frac{6}{1000} R$

(See page 62)

* A sure way to avoid errors would be to use both these methods.

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And

And by writing for P , Q , and R , their values in terms of α , β , &c. as expressed by the equations in page 61, we have the following very commodious *formulae* for calculating the several orders of differences.

1st Differences of Logarithms.	2d Difference.	3d Differences.
$\frac{1}{10}\alpha + (\frac{1}{20} - \frac{1}{200})\beta + (\frac{1}{6000} - \frac{1}{60})\gamma$	$\frac{1}{100}\beta$	$\frac{1}{1000}\gamma$
$\frac{1}{10}\alpha + (\frac{1}{20} + \frac{1}{200})\beta + (\frac{1}{6000} - \frac{1}{60})\gamma$		$\frac{1}{1000}\gamma$

Or,

1st Differences of Logarithms.	2d Difference.	3d Differences.
$\frac{1}{10}\alpha' - (\frac{1}{20} + \frac{1}{200})\beta + (\frac{1}{6000} - \frac{1}{60})\gamma$	$\frac{1}{100}\beta$	$\frac{1}{1000}\gamma$
$\frac{1}{10}\alpha' - (\frac{1}{20} - \frac{1}{200})\beta + (\frac{1}{6000} - \frac{1}{60})\gamma$		$\frac{1}{1000}\gamma$

Now, for an example of the use of the above *formulae*, let us take the differences of the logarithms of the five arithmetic progressionals whose mean is 5000, and common difference 1; then will

$$\alpha = 0.00008685021164,9$$

$$\beta = 0.00000001737178,0$$

$$\gamma = 0.00000000000695,1$$

which values of α , β , and γ being written in their proper places, we have

0.00000868580278, 04	17371, 780	,6951
1st Difference of Logarithms.	2d Difference.	3d Difference.

For a second example of the use of these *formulae*, let us take the differences of the logarithms of the five progressions whose mean is 5001, and common difference 1; then will

$$\alpha' = 0.00008685021164, 9$$

$$\beta = 0.00000001736483, 3$$

$$\gamma = 0.0000000000694, 7$$

From which we get

0.00000868406598, 45	17364, 833	,6947
1st Difference of Logarithms.	2d Difference.	3d Difference.

REMARKS.

Here it is obvious, that it is not necessary to calculate, by the above *Formulae*, as many sets of the several orders of differences as there are numbers in the progression of which the extremes are 9090 and 5000 and

Note. In filling up the table, the mean of the five arithmetic progressions, and consequently its logarithm, is always to be written in the same line with the second difference obtained by the above *formulae*.

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common

common difference 1. For it appears that the third differences near this end of the table, where they vary most from each other, are so nearly equal, that either of them may be used for the other, for a greater interval than that between them, without occasioning any error in the logarithms.

With what facility a table of logarithms may be constructed by the method above described, I leave to the determination of those, who will make trial of different methods, or are competent judges of what has been done in this Essay.

The logarithms of every 9th, 8th, 7th, 6th, 5th, 4th, and 3d number, within proper limits, and of every 2d number from 66666 to 50000, may be obtained by a method similar to that by which the logarithm of every 10th number from 90900 to 50000 was derived. These logarithms may, perhaps, on some occasions, be useful checks; but the easiest case of interpolation is, when x , as above, is $= \frac{1}{10}$.

Upon a review of what has been done in this Essay, it appears, that the chief use of the series and approximations given in the former part of it, is, to find any difference, to the 4th order, independent of any logarithms or other differences whatever; and to derive one difference from another: And that the advantage

vantage of the *Newtonian method* of interpolation consists in easily finding a set of the several orders of differences, for the purpose of filling up an interval of logarithms, when several logarithms, and consequently their differences, are given.

The facility of constructing a table by the joint use of these methods may possibly invite some gentleman who has leisure, to compute a table of hyperbolic logarithms to nine or ten places of figures besides the index, for the numbers 10.000, 9.999, 9.998, &c. to 1.000; a work which would be highly acceptable to mathematicians in general, and to fluxionists in particular.

In computing this new table some advantage might be made of the old one, which contains the hyperbolic logarithms to seven places of figures besides the index, of the numbers 10.00, 9.99, 9.98, &c. to 1.00. Of this I will give an instance, in easily deriving the first of the third differences wanted in the computation of the new table, from the first of the first differences at the end of the old table. This first difference taken from the old table is 0.0010005, which call α ; then, by the second equation for computing third differences from first differences by division, given in page 37, we shall have $2\alpha \div : a a - 2a + \frac{1}{3}$, &c. = γ' ; where $a = 999$. Now it will be sufficient for our purpose to use only the first two terms of the divisor, which

which may be readily taken out of a table of square numbers, (to be found among Dr. *Hutton's* Tables of powers and products *), and then the arithmetical work will stand as below.

$$\begin{array}{r} aa = 998001 \\ - 2a = - 1998 \\ \hline 996003 \end{array} \quad 0.0010005 = a \quad (0.00000000200903 = \gamma;$$

the first of the third differences. And if this third difference be multiplied by 3 and divided by 996.5, according to the first equation for calculating fourth differences from third differences by division, given in page 38, the quotient will be 0.00000000006048, which is the first fourth difference; *i. e.* it is the fourth difference of the hyperbolic logarithms of five numbers in arithmetic progression, of which the mean is 998 and common difference 1. These instances of the utility and easy operation of the series given, page 34 to 38, for calculating one difference from another, cannot be easily overlooked, even by an inattentive reader: and many others might be produced.

With regard to examining a table of logarithms, or any part of it, by successively taking the differences, provided that all the logarithms were set down true to the nearest unit in their lowest place of decimals,

* This book is on so large a plan, and of such extensive utility, that it wants only to be generally known to be generally used.

the following rules (evident to any one who considers the forms of the several orders of differences as they stand in page 22) are to be observed.

I. When the first order only, of differences, is found in the table, or, in other words, when the true second differences do not affect the last figure of the logarithms, (then, since no logarithm ought to differ from the truth by more than half an unit in the lowest place of decimals) no two contiguous first differences ought to vary from each other by more than 2 in the lowest place of figures.

II. When both the first and second orders of differences enter the table, but the third vanishes, no two contiguous second differences ought to differ from each other by more than 4.

III. When the first, second, and third orders of differences are found in the table, but the fourth vanishes, the greatest difference of any two contiguous third differences ought not to exceed 8.

It is to be further remarked, that (under the restrictions specified in these rules,) the chance is against the happening of so great a difference as 2 between any two contiguous first differences taken at random; that the chances against the happening of so great a variation from each other as 4, in two contiguous second differences, so taken, are several; and still

more

more are against the difference of 8 between two third differences.

Of my other labours on this subject something may probably appear hereafter, if I should find leisure to resume it.

E S S A Y III.

*On the Reduction of Equations that have two equal Roots :
containing an Investigation of the common Method, together
with some Remarks.*

IT has long been known, and Mr. *McLaurin* has in a masterly manner demonstrated, that, when an equation has two roots equal to each other, it may be depressed to a lower dimension; *i. e.* another equation may be easily formed from it, in which the index of the highest power of the unknown quantity shall be less by 1 than it was in the original equation. Thus, if the given equation be a quadratic, one of its equal roots may be had from a simple equation; if the given equation be a cubic, one of the equal roots may be had from a quadratic, &c. How this method came to be at first discovered it is not my present business to enquire: perhaps a notation similar to that which follows might lead to it. But be that as it may, the following investigation of the common method of depressing such equations is different from any other that has come to my hands; and as it is such as experience has proved to be well adapted to the capacities

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of learners, I take this opportunity of presenting it to the public.

THEOREM I.

If the roots of the quadratic equation $xx - 2px + q = 0$ are equal to each other x , will be $= \frac{1}{2}p$.

INVESTIGATION.

Let each of the values of x be $= a$; then, by multiplying these two simple equations $x - a = 0$, and $x - a = 0$, by each other, there will be had $xx - 2ax + aa = 0$, of which equation every term is equal to its corresponding one in the original equation: thus, $xx = xx$, $2ax = px$, and $aa = q$. Now, since $2ax = px$, by dividing by $2x$ we have $a = \frac{1}{2}p$. But a is $= x$; therefore $x = \frac{1}{2}p$. Q. E. I.

COROLLARY I. In the above investigation we have $aa = q$; therefore $a = \sqrt{q}$, which is another expression of the value of x . But, as the division of a quantity by 2 is in general an easier operation than extracting the square root, the former is to be preferred.

COROL. II. Since x is $= \frac{1}{2}p$, and $= \sqrt{q}$ also, it follows, that, when any quadratic equation has equal roots, $\frac{1}{2}p = \sqrt{q}$, or $\frac{1}{2}pp = q$, which is proved also by the common method of completing the square.

EXAMPLE.

EXAMPLE. Let the equation given be $xx - 10x + 25 = 0$; then will $x = \frac{10}{2} = \sqrt{25} = 5$.

THEOREM II.

If the cubic equation $x^3 - px^2 + qx - r = 0$ has two equal roots, one of them will be a root of this equation $3x^2 - 2px + q = 0$.

INVESTIGATION.

Conceive the equation to be formed by the multiplication of these two, $xx - 2ax + aa = 0$, & $x - \pi = 0$; then shall we have

$$\left. \begin{array}{l} x^3 - 2ax \\ - \pi \end{array} \right\} \left. \begin{array}{l} x^2 + aa \\ + 2a\pi \end{array} \right\} x - aa\pi = x^3 - px^2 + qx - r = 0;$$

and by putting the homologous terms equal to each other, there will be had

$$1. \quad 2a + \pi = p$$

$$2. \quad aa + 2a\pi = q$$

$$3. \quad aa\pi = r. \quad \text{From the first}$$

of these equations we have $\pi = p - 2a$, which being written for it in the second we have $aa + 2a(p - 2a) = q$, or $3aa - 2pa + q = 0$. And since $a = x$, we have $3xx - 2px + q = 0$. Q. E. I.

COROL. I. By writing $p - 2a$ for π in the third equation, we have $aa(p - 2a) = r$, or $2a^3 - pa^2 + r = 0$;

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and

and x being written for a , this equation becomes
 $2x^3 - px^2 + r = 0$.

COROL. II. When $p = 0$, the quadratic equation in the theorem becomes $3xx + q = 0$, and $x = \sqrt{-\frac{1}{3}q}$.

COROL. III. When $p = 0$, the equation in Corol. I. becomes $2x^3 + r = 0$, from which we have $x = \sqrt[3]{-\frac{1}{2}r}$.

COROL. IV. Therefore, when the cubic equation $x^3 + qx - r = 0$ has two equal roots, $(x =) \sqrt{-\frac{1}{3}q} = \sqrt[3]{-\frac{1}{2}r}$, or $-4q^3 = 27r^2$.

COROL. V. When $q = 0$, $3xx - 2px = 0$, & $x = \frac{2}{3}p$.

COROL. VI. When $q = 0$, $aa + 2a\pi = 0$, & π , the unequal root, becomes $= -\frac{1}{2}a$.

COROL. VII. Therefore, when $q = 0$, $aa\pi = -\frac{1}{2}a^3 = r$; and $a (= x) = \sqrt[3]{-2r}$.

COROL. VIII. Therefore, when the equation $x^3 - px^2 - r = 0$ has two equal roots, $\frac{2}{3}p = \sqrt[3]{-2r}$, or $4p^3 = -27r$.

COROL. IX. When $r = 0$, $\pi = 0$; and then, as the original equation becomes a quadratic, so the equation in Corol. I. becomes $2x^2 - px = 0$, or $2x - p = 0$, & $x = \frac{1}{2}p$, which perfectly agrees with Theorem I.

COROL. X. When $r = 0$, as the original equation becomes $x^2 - px + q = 0$, it appears that one of the equal roots of this equation is also a root of the quadratic $3x^2 - 2px + q = 0$.

SCHOLIUM

SCHOLIUM I. Although the last equation be very different from that in the preceding Corol. yet one value of x is common to both these equations. This may be proved different ways; but a very obvious one is to take $x^2 - px + q = 0$ from $3x^2 - 2px + q = 0$, and the remainder will be $2x^2 - px = 0$, which is the very equation obtained in Corol. 9.

SCHOL. II. By means of the equations $-4q^3 = 27r^2$, and $4p^3 = -27r$, (Corol. 4 and 8) and a table of square and cube numbers, it may quickly be discovered whether the cubic equations $x^3 + qx - r = 0$, and $x^3 - px - r = 0$, have equal roots or not. But here it should not be hastily concluded that, when both p and q are found in a cubic equation, the easiest method to discover whether it has equal roots or not is to transform it into another which shall want the second term, and then see whether $-4q^3 = 27r^2$. For this case easier methods will be given in the next Essay.

EXAMPLE I. Let the equation given be $x^3 + 5x^2 - 32x + 36 = 0$, which has equal roots; then, by the Theorem, $3x^2 + 10x - 32 = 0$, whose roots are 2, & $-\frac{16}{3}$, the former of which is one of the equal values of x in the given cubic equation.

EXAMPLE

EXAMPLE II. Let the equation be $x^3 - 27x + 54 = 0$; then, by Corol. 2, one of the equal values of x is $\sqrt[3]{9} = 3$. By Corol. 3, $x = \sqrt[3]{27} = 3$.

EXAMPLE III. Given $x^3 + \frac{10}{7}x^2 - \frac{4000}{9261} = 0$, to find the equal values of x .

By Corol. 5, we have $x = -\frac{20}{31}$.

THEOREM III.

If the biquadratic equation $x^4 - px^3 + qx^2 - rx + s = 0$ has two equal roots, one of those roots may be had from this equation, $4x^3 - 3px^2 + 2qx - r = 0$.

INVESTIGATION.

Let the given equation be the product of these two quadratic equations, $x^2 - \pi x + \rho = 0$, and $x^2 - 2ax + aa = 0$; then will

$$\left. \begin{array}{l} x^4 - \pi x^3 + \rho x^2 \\ - 2ax^3 + 2a\pi x^2 - 2a\rho x \\ + aa x^2 - aa\pi x + aa\rho \end{array} \right\} = x^4 - px^3 + qx^2 - rx + s = 0;$$

and by putting the respective terms equal to each other, we have

$$\begin{array}{ll} 1. & 2a + \pi = p. \\ 2. & aa + 2a\pi + \rho = q. \\ 3. & aa\pi + 2a\rho = r. \\ 4. & aa\rho = s. \end{array}$$

Now,

Now, by writing $-2a + p$ for π in the second and third equations, we get

$$\left. \begin{array}{l} aa \\ -4aa + 2ap \\ + \epsilon \end{array} \right\} = q, \text{ or, } \epsilon = 3aa - 2pa + q.$$

$$\left. \begin{array}{l} -2a^2 + paa \\ 6a^3 - 4paa + 2qa \end{array} \right\} = r, \text{ or, } 4a^3 - 3pa^2 + 2qa = r.$$

And by writing x for a , and transposing r to the other side of the equation, we have $4x^3 - 3px^2 + 2qx - r = 0$.

Q. E. I.

COROLLARY I. By writing $3aa - 2pa + q$ for ϵ in the fourth equation, we get $3a^4 - 2pa^3 + qa^2 = s$; which equation, x being written for a , and s transposed to the other side, becomes $3x^4 - 2px^3 + qx^2 - s = 0$.

COROL. II. When $p = 0$, the last equation becomes of a quadratical from $3x^4 + qx^2 - s = 0$.

COROL. III. When $q = 0$, we have (by the Theorem and Corol. I.) these two equations, $4x^3 - 3px^2 - r = 0$, and $3x^4 - 2px^3 - s = 0$; and if the former of these multiplied by p be added to the latter multiplied by 2, the sum will be $6x^4 - 3ppx^2 - pr - 2s = 0$, another quadratic.

COROL.

COROL. IV. When $r=0$, the cubic equation in our Theorem becomes $4x^2 - 3px + 2q = 0$.

COROL. V. When $s=0$, $t=0$; and then, as the original equation becomes a cubic, so the equation in Corol. I. becomes $3x^2 - 2px + q = 0$, which is the same as that in Theorem 2.

EXAMPLE I. Let the given biquadratic equation which has equal roots be $x^4 - 9x^2 + 4x + 12 = 0$; then, by the Theorem, $4x^3 - 18x + 4 = 0$, from which one of the equal roots comes out $= 2$. By Corol. 2, we have $3x^4 - 9x^2 - 12 = 0$, or $x^4 - 3x^2 - 4 = 0$, in which equation one value of x is 2.

EXAMPLE II. Let the equation $x^4 - 2x^3 - 54x + 135 = 0$ be given, to find its equal roots.

By the Theorem, $4x^3 - 6x^2 - 54 = 0$, one of whose roots is 3. By Corol. 3, $6x^4 - 12x^2 - 378 = 0$, or $x^4 - 2x^2 - 63 = 0$, in which equation one value of x is also 3.

THEOREM IV.

If the sursolid equation $x^5 - px^4 + qx^3 - rx^2 + sx - t = 0$ has two roots equal to each other, one of those equal roots shall be a root also of this equation, $5x^4 - 4px^3 + 3qx^2 - 2rx + s = 0$.

I N V E S -

INVESTIGATION.

Let the fursolid equation be the product of the quadratic $x^2 - 2ax + aa = 0$, and cubic $x^3 - \pi x^2 + \rho x - \sigma = 0$; then shall

$$\left. \begin{aligned} x^5 - \pi x^4 + \rho x^3 - \sigma x^2 \\ - 2ax^4 + 2a\pi x^3 - 2a\rho x^2 + 2a\sigma x \\ + aa x^3 - aa\pi x^2 + aa\rho x - aa\sigma \end{aligned} \right\} =$$

$x^5 - px^4 + qx^3 - rx^2 + sx - t$. Here again, by equating the homologous terms, we get

$$1. \quad 2a + \pi = p.$$

$$2. \quad aa + 2a\pi + \rho = q.$$

$$3. \quad aa\pi + 2a\rho + \sigma = r.$$

$$4. \quad aa\rho + 2a\sigma = s.$$

$$5. \quad aa\sigma = t. \quad \text{And from these}$$

equations, by transposition and substitution, we get

$$\pi = -2a + p.$$

$$\left. \begin{aligned} aa \\ - 4aa + 2ap \\ + \rho \end{aligned} \right\} = q, \text{ or } \rho = 3aa - 2pa + q.$$

$$\left. \begin{aligned} - 2a^3 + pa^2 \\ 6a^3 - 4pa^2 + 2qa \\ + \sigma \end{aligned} \right\} = r, \text{ or } \sigma = -4a^3 + 3pa^2 - 2qa + r$$

$$\left. \begin{aligned} 3a^4 - 2pa^3 + qa^2 \\ - 8a^4 + 6pa^3 - 4qa^2 + 2ra \end{aligned} \right\} = s, \text{ or } 0 = 5a^4 - 4pa^3 + 3qa^2 - 2ra + s.$$

M

And

And by writing x for a we have $5x^4 - 4px^3 + 3qx^2 - 2rx + s = 0$. *Q. E. I.*

COROLLARY I. By writing x for a in the 5th equation above, and transposing $aa\sigma$, we obtain $(t - aa\sigma = 0) \quad 4x^5 - 3px^4 + 2qx^3 - rx^2 * + t = 0$.

COROL. II. When $t = 0, \sigma = 0$; and then the sur-solid equation becomes a biquadratic, and the last equation becomes $4x^3 - 3px^2 + 2qx - r = 0$, which is the same as that in Theorem 3.

COROL. III. It follows also that, when $t = 0$, one of the equal roots of the biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$ is a root of this biquadratic $5x^4 - 4px^3 + 3qx^2 - 2rx + s = 0$.

SCHOLIUM I.

By putting both s and $t = 0$, we shall obtain two cubic equations, whose terms are multiplied by different arithmetical progressions, in each of which one value of x shall be one of the equal values of it in the original cubic. And by putting r, s , and t , each $= 0$, we shall have two different quadratics, of each of which one root shall be one of the equal roots of the given quadratic. But it is very easy to demonstrate (as Mr. McLaurin has done) that, if the terms of any one of the

the above equations that has equal roots be multiplied by any arithmetical progression whatever, one of the roots of this new equation shall be one of the equal roots of the original equation. For instance, let us take the two equations in the last Theorem, and multiply the former by m and the latter by nx , and there will be

$mx^5 - mpx^4 + mqx^3 - mrx^2 + msx - mt = 0$,
and $5nx^5 - 4npx^4 + 3nqx^3 - 2nrx^2 + nsx = 0$; and by taking the sum and difference of these equations we shall have

$$\overline{m+5n.x^5} - \overline{m+4n.px^4} + \overline{m+3n.qx^3} - \overline{m+2n.rx^2} + \overline{m+n.sx} - mt = 0,$$

and $\overline{m-5n.x^5} - \overline{m-4n.px^4} + \overline{m-3n.qx^3} - \overline{m-2n.rx} + \overline{m-n.sx} - mt = 0;$

where it is evident that m and n may be any numbers whatever, and that any term of this progression may be put equal to 0; so that we may destroy any term we please of the original equation.

Thus, the terms of the cubic equation $x^3 - px^2 + qx - r = 0$ being multiplied by the progression

$$\left. \begin{array}{cccc} 3, & 2, & 1, & 0 \\ 2, & 1, & 0, & -1 \\ 1, & 0, & -1, & -2 \\ 0, & -1, & -2, & -3 \end{array} \right\} \text{we have} \left\{ \begin{array}{l} 3x^3 - 2px^2 + qx - * = 0, \\ 2x^3 - px^2 - * + r = 0. \\ x^3 - * - qx + 2r = 0. \\ * + px^2 - 2qx + 3r = 0. \end{array} \right.$$

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In

In all which equations one value of x is the same as one of its equal values in the original cubic equation.

S C H O L I U M II.

In the investigation of Theorem 4, the law of continuation is manifest; at present, however, I shall not apply it to any higher equation, but give an example of the use of the Theorem.

E X A M P L E.

Let the equation given be $x^5 + x^3 - x^2 + 0.09433 = 0$; then, shall $5x^3 + 3x - 2 = 0$, from which x comes out $= 0.4811$ nearly, which is one of the equal roots of the given sursolid equation.

P O S T S C R I P T.

By a notation similar to that used in this Essay, theorems for reducing equations that have three equal roots, may easily be investigated. And the variety of equations attainable this way is not inconsiderable.

E S S A Y IV.

*On the Resolution of Equations that have two equal Roots:
containing some new Theorems for calculating the Values of
those Roots.*

IN the preceding Essay, (and in other places) it has been demonstrated, that, when an equation has two equal roots, it may very easily be reduced to a lower dimension. In this Essay it will appear, that such equations are easily reducible to any dimension desired, even to a single one.

The following Theorems are indeed some of my most early productions, and appear in the state in which they were at first investigated, and in which they were presented to one of the most learned Societies * in Europe. The approbation this paper (as well as a former one) met with from that Philosophic Body, and, since it was printed in their Transactions,

* The *Royal Society* of London, before whom my paper was read, June 20, 1782. It is reprinted here without any material alteration, and with but few additions.

from

from some learned foreigners *, have been no small inducements with me to publish this collection; and cannot but give me ground to hope, that my labours will prove in some measure useful to the public.

THEOREM I.

If the cubic equation $x^3 - px^2 + qx - r = 0$ has two equal roots, each of them will be $(x) = \frac{pq - 9r}{2pp - 6q}$.

DEMONSTRATION.

Call the three roots a, a , and b ; then, by the composition of equations, we shall have $x^3 - 2a \} x^2 + aa \} x -$
 $aab = 0$, where $2a + b = p$, $aa + 2ab = q$, and $aab = r$; which values being written in our Theorem, we have
 $x \left(= \frac{pq - 9r}{2pp - 6q} \right) = \frac{2aaa + 4aab + aab + 2abb - 9aab}{8aa + 8ab + 2bb - 6aa - 12ab} = \frac{2aaa - 4aab + 2abb}{2aa - 4ab + 2bb} = a$.
 Q. E. D.

COROLLARY I. Hence, and by Theorem 2, of the preceding Essay, we may easily find whether any cubic equation has equal roots or not. For, taking $x = \frac{pq - 9r}{2pp - 6q}$, if $3xx - 2px + q$ be $= 0$, the equation has equal roots, otherwise it has not.

* See the Rev. Dr. Hales's *Analysis Æquationum*, page 179.

COROL.

COROL. II. From the last equation, by writing $\frac{pq - 9r}{2pp - 6q}$ for x , we easily obtain this, $\overline{qq - 4pr} \times pp + 18pr - 4qq \times q = 27rr$, by which also it may quickly be known whether any cubic equation has equal roots or not.

COROL. III. When $p = 0$, the last equation becomes $-4q^3 = 27rr$, which is the same as that in Corol. 4, page 76.

COROL. IV. When $q = 0$, we have $-4p^3 = 27r$, which agrees with Corol. 8, page 76.

EXAMPLE I. Let the equation given be $x^3 + 5x^2 - 32x + 36 = 0$, which has two equal roots; it is proposed to find them by the above Theorem.

Here $p = -5$, $q = -32$, and $r = -36$; these values being written in the theorem, we have $\frac{-5 \times -32 - 9 \times -36}{2 \times 25 - 6 \times -32} = \frac{160 + 324}{50 + 192} = \frac{484}{242} = 2$, which is one of the equal roots.

And if, in proof of the work, 2 be written for x , the equation becomes $8 + 20 - 64 + 36$, which is evidently $= 0$. Otherwise, dividing the given cubic by the quadratic $\overline{x - 2}^2$, we have $x^2 - 4x + 4$) $x^3 + 5x^2 - 32x + 36$ ($x + 9$; therefore the three roots are 2, 2, and -9 .

Had

Had the question been to find whether the cubic equation $x^3 + 5x^2 - 32x + 36 = 0$ has equal roots or not, it might have been very easily answered. For by the first Corollary, 2, the value of x given by the Theorem, being written for it in the quadratic equation $3x^2 + 10x - 32 = 0$, the result is $12 + 20 - 32 = 0$; therefore the proposed equation has equal roots. The same thing might have been quickly discovered by Corol. 2; the quantities on the first side of the equation there given being $7600 + 27392$, and the quantity on the second 34992 , which is equal to their sum.

EXAMPLE II. Let it be required to find the equal roots of this equation, $x^3 + 27x + 54 = 0$.

Here $p = 0$, and the value of x given by the Theorem is $\frac{-3 \times -54}{-2 \times -27} = 3$, which is one of the equal roots sought.

EXAMPLE III. Given $x^3 + \frac{10}{7}x^2 - \frac{4000}{9261} = 0$, an equation which has equal roots, to find them.

Here $q = 0$, and the Theorem gives $\frac{-36000 \times 49}{290 \times 9261} = \frac{-20}{21}$, which value being written for x the equation vanishes.

THEOREM

THEOREM II.

If the biquadratic equation $x^4 - px^3 + qx^2 - rx + s = 0$ has two equal roots, make $A = \frac{12r - 2pq}{3pp - 8q}$, $B = \frac{pr - 16s}{3pp - 8q}$, $C = \frac{4B - 2q}{4A + 3p}$, and $D = \frac{r}{4A + 3p}$, and you will have $x = \frac{D - B}{A - C}$.

A synthetical demonstration of this Theorem would be very long: the INVESTIGATION is as follows.

It has been demonstrated by the writers on algebra, that, if a biquadratic equation, as $x^4 - px^3 + qx^2 - rx + s = 0$, has two equal roots, one of them may be had from the equation $4x^3 - 3px^2 + 2qx - r = 0$. Multiply this equation by x , and the original one by 4, and take the difference of the two, which will be $px^3 - 2qx^2 + 3rx - 4s = 0$. Again, if this equation be multiplied by 4, and the other cubic by p , and their difference taken, we shall have $3pp - 8q \times x^2 + 12r - 2pq \times x + pr - 16s = 0$, or $x^2 + \frac{12r - 2pq}{3pp - 8q} x + \frac{pr - 16s}{3pp - 8q} = 0$, or $x^2 + Ax + B = 0$, putting A and B for the known quantities in the second and third terms. Now multiply this equation by $4x$, and take the first cubic from it, and we shall have $4A + 3p \times x^2 + 4B - 2q \times x + r = 0$, which being divided by $4A + 3p$, and C and D put

N

equal

equal to $\frac{4B-2q}{4A+3p}$ and $\frac{r}{4A+3p}$ respectively, gives $x^2 + Cx + D = 0$; and this equation being taken from the other quadratic, there remains $\overline{A-C} \times x + B-D = 0$; consequently $x = \frac{D-B}{A-C}$. Q. E. I.

COROLLARY I. From the above investigation it appears, that one of the equal roots may also be obtained from either of these two quadratic equations, of which the first seems most eligible, as the co-efficients of it are less complex than those of the other:

$$\overline{3pp-8q} \times x^2 + \overline{12r-2pq} \times x + pr-16s = 0,$$

and $\overline{4A+3p} \times x^2 + \overline{4B-2q} \times x + r = 0$. And these, when $p=0$, become $-8qx^2 + 12rx - 16s = 0$,

$$\text{and } -\frac{48r}{8q}x^2 + \frac{64s}{8q} - 2q \times x + r = 0;$$

$$\text{or } x^2 - \frac{3r}{2q}x + \frac{2s}{q} = 0,$$

$$\text{and } x^2 + \frac{qq-4s}{3r}x - \frac{q}{6} = 0.$$

COROL. II. When $q=0$, the two quadratics become

$$3ppx^2 + 12rx + pr - 16s = 0,$$

and $\frac{16r+3p^3}{pp}x^2 + \frac{4pr-64s}{3pp}x + r = 0,$

$$\text{or } x^2 + \frac{4r}{pp}x + \frac{pr-16s}{3pp} = 0,$$

$$\text{and } x^2 + \frac{4pr-64s}{48r+9p^3}x + \frac{ppr}{16r+3p^3} = 0.$$

COROL.

COROL. III. If both p and q vanish, then, from either of the quadratics we get $x = \frac{4s}{3r}$, perfectly agreeing with the cubic $px^3 - 2qx^2 + 3rx - 4s = 0$, which, when p and q vanish, becomes $3rx - 4s = 0$. And this equation is of use; because, in this case, the theorem fails, one of the divisors being $= 0$.

COROL. IV. From the equation $4x^3 - 3px^2 + 2qx - r = 0$, which, when p and q vanish, becomes $4x^3 - r = 0$, we also get $x = \sqrt[3]{\frac{r}{4}}$, another expression of the same value of x .

COROL. V. When $r = 0$, $D = 0$, and from the equation $x^2 + Cx + D = 0$, we have $x = -C$.

EXAMPLE I. If the equation $x^4 - 9x^3 + 4x + 12 = 0$ has equal roots, it is proposed to find them.

Here $p = 0$, $q = -9$, $r = -4$, and $s = 12$; and

$$A \text{ becomes } = \frac{-2}{3},$$

$$B \quad . \quad . \quad = \frac{-8}{3},$$

$$C \quad . \quad . \quad = \frac{-11}{4},$$

$$D \quad . \quad . \quad = \frac{3}{2},$$

$$\text{and } \frac{D-B}{A-C} = \frac{18+32}{-8+33} = \frac{50}{25} = 2,$$

N 2

which,

which being written for x , the equation becomes $16 - 36 + 8 + 12 = 0$; therefore 2 is one of the equal roots.

The same value of x may be discovered from either of the quadratic equations mentioned in Corollary 1. The proper values of the co-efficients being written in the first of them, it becomes $x^2 - \frac{2}{3}x - \frac{8}{3} = 0$, where one value of x is $\frac{1 + \sqrt{25}}{3} = 2$. The other quadratic becomes $x^2 - \frac{11}{4}x + \frac{3}{2} = 0$, one of whose roots is $\frac{11 + \sqrt{25}}{8} = 2$.

EXAMPLE II. It being known that the equation $x^4 - x^3 - 7x^2 + 13x - 6 = 0$ has two equal roots, to find them.

Here $p = 1$, $q = -7$, $r = -13$, and $s = -6$; and $A = \frac{-142}{59}$, $B = \frac{83}{59}$, $C = \frac{-1158}{391}$, $D = \frac{767}{391}$, $D - B = \frac{12800}{23069}$, $A - C = \frac{12800}{23069}$, and $\frac{D-B}{A-C} = \frac{12800}{12800} = 1$, one of the roots sought.

The same value of x may be found from either of the two general quadratic equations given in Corollary 1. From the first of them we get one value of $x = \frac{71 - \sqrt{144}}{59} = 1$. And from the other, one value $= \frac{579 - \sqrt{35344}}{391}$, which is also $= 1$.

EXAMPLE

EXAMPLE III: Let the equation $x^4 - 2x^3 * - 54x + 135 = 0$ be given, to find its equal roots.

By Corol. 2. we have these two quadratic equations, $x^2 + 54x - 171 = 0$, and $x^2 - \frac{114}{37}x + \frac{9}{37} = 0$; in the former of which one value of x is $-27 + \sqrt{900} = 3$; in the latter one value of x is $\frac{57 + \sqrt{2916}}{37}$, which is also $= 3$: and these are the roots fought *.

EXAMPLE IV. Given the equation $x^4 - \frac{1}{2}x + \frac{3}{16} = 0$, in which two values of x are equal to each other, to find them.

By Corollary 3, we have $x = \frac{4 \times 3}{16} \div \frac{3 \times 1}{2} = \frac{8}{16} = \frac{1}{2}$. By Corol. 4, x is $= \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$.

THEOREM III.

If the sursolid equation $x^5 - px^4 + qx^3 - rx^2 + sx - t = 0$ has two roots equal to each other, and you make $A = \frac{15r - 3pq}{4pp - 10q}$, $B = \frac{2pr - 2os}{4pp - 10q}$, $C = \frac{25t - ps}{4pp - 10q}$, $D = \frac{5B - 3q}{5A + 4p}$, $E = \frac{5C + 2r}{5A + 4p}$, $F = \frac{s}{5A + 4p}$, $G = \frac{B - E}{A - D}$, $H = \frac{F + C}{A - D}$, $I = \frac{B - H}{A - G}$, and $K = \frac{C}{A - G}$, then shall one of the equal values of x be $= \frac{H - K}{I - G}$,

* By comparing this with Example 2, page 80, the agreement of the results of different theorems will appear.

The

The investigation of this theorem being altogether similar to that of the last, it is omitted here.

SCHOLIUM I. The difference of equations being taken step by step, as in the investigation of Theorem 2, it will appear, that one of the equal roots may also be had from any one of the following five equations, of which sometimes one, sometimes another, will be the most eligible.

$$1. \ 5x^4 - 4px^3 + 3qx^2 - 2rx + s = 0.$$

$$2. \ px^4 - 2qx^3 + 3rx^2 - 4sx + 5t = 0.$$

$$3. \ x^3 + Ax^2 + Bx + C = 0.$$

$$4. \ x^3 + Dx^2 + Ex - F = 0.$$

$$5. \ x^2 + Gx + H = 0.$$

SCHOLIUM II. It is obvious, that, when p vanishes, the work will be considerably shortened; and when both p and q are wanting, though the above *formula* fails, yet the equal root may be easily obtained from the equation $px^4 - 2qx^3 + 3rx^2 - 4sx + 5t = 0$, which in that case becomes $3rx^2 - 4sx + 5t = 0$.

Whenever s is wanting, F , in the second cubic above, will be $= 0$, and consequently x may be found from the quadratic equation $x^2 + Dx + E = 0$.

But

But in any of these cases the equal root may be found by division. However, the operation probably will not, in general, be so short as extracting the root of the quadratic.

I will now give an example or two, by way of illustration.

EXAMPLE 1. Given $x^5 + x^3 - x^2 + 0.09433 = 0$, to find x , two values of it being equal to each other.

Here $p = 0$, $q = 1$, $r = 1$, $s = 0$, $t = -0.09433$, and we get

$A = -1.5$	$F = 0$
$B = 0$	$G = -0.22310$
$C = +0.23583$	$H = -0.12412$
$D = \pm 0.4$	$I = -0.09720$
$E = -0.42388$	$K = -0.18469$

$$\text{and } x = \frac{H-K}{I-G} = 0.4811 \text{ nearly.}$$

The proper values of the co-efficients being written in the five equations given above, and some of them divided by the co-efficient of the highest power of x , we have these four equations, in each of which one value of x is one of the equal ones sought:

$$x^3 * + 0.6x - 0.4 = 0.$$

$$x^3 - 1.5x^2 * + 0.23583 = 0.$$

$$x^2 + 0.4x - 0.42388 = 0.$$

$$x^2 - 0.2231x - 0.12412 = 0.$$

Now the most eligible equation is the quadratic $x^2 + 0.4x - 0.42388 = 0$, whose affirmative root is $\sqrt{0.46388} - 0.2 = 0.48109$ very nearly, agreeing with the value of x found above, but true to one place lower in the decimal.

EXAMPLE II. To find the two equal values of x , in the equation $64x^5 - 20x^2 + 3 = 0$.

The given equation being divided by 64, we have $x^5 - 0.3125x^2 + 0.046875 = 0$; and then, from the first of the five equations given in Schol. I, we get $5x^4 - 0.625x = 0$, and $x = \sqrt[3]{0.125} = 0.5$. But from the second of the equations just now mentioned, we have $0.9375x^2 - 0.234375 = 0$, or $x^2 = \frac{0.234375}{0.9375} = 0.25$, and $x = \sqrt{0.25} = 0.5$.

GENERAL SCHOLIUM.

The method which has now been explained, of computing the values of the equal roots, is evidently applicable to equations of all dimensions; at present, however,

however, I shall not apply it to any equation of more than five dimensions, for two reasons: first, because such equations occur less frequently than those of lower powers: and secondly, because the theorems will be very complex.

I beg leave to add, that this and the preceding Essay, are but two sections of *A New System of Algebra*, planned and begun several years ago, in which I intended to treat distinctly of equations that have two, as well as those that have three, equal roots; and to apply those equations to some new uses, at least such as I have not hitherto been able to find in any other book. When I shall have leisure and opportunity to finish that work is uncertain. In the mean while, I hope, these little pieces will be candidly received by those who have more leisure, better abilities, and more inducements to studies of this kind.

E S S A Y V.

On the Correction of Fluents found by Descending Series.

ALTHOUGH the finding of fluents by descending series has been often mentioned by the writers on fluxions, yet that method does not appear to have been brought into use, in the solution of problems, even by those late and celebrated writers *Emerson* and *Simpson*, who, in their treatises of fluxions, have given no instances of the actual use of such series. Nor is there, to the best of my recollection, an instance of it in the *Ladies' Diary*, which may justly be considered as the barometer of mathematics in this island.

This neglect of descending series is the more remarkable, as their convergency, by the powers of the flowing quantity, always begins when the convergency of the ascending series, by the powers of the same quantity, ceases.

These observations, (too evident to escape any attentive learner,) occurred to me early in my study of fluxions, and occasioned various conjectures concerning
ing

ing the cause of the neglect of descending series. After some time, however, recollecting that I had seen nothing advanced against the usefulness of that kind of series, I thought of the experiment of computing the values of some fluents taken in descending series, in cases where the convergency was swift, and comparing those values with the values of the correct fluents taken in other terms. And by this experiment I discovered, that *fluents taken in descending series commonly want a correction, which being applied, those series become no less useful than ascending ones: the establishment of which truth is the design of this Essay.*

The calculation of this correction, indeed, is not always easy: sometimes, however, it is obtained with great facility. Of both these cases some instances will appear in the subsequent pages.

Here it may be proper to mention, that, after materials for this Essay were collected, the learned Mr. EULER's *Institutiones Calculi Integralis* happening to come into my hands, I found that the difference of the values given by the two kinds of series in question had been observed by him. He says, (vol. I. page 105) "*Inde autem, quod duas series pro z exhibemus, minime sequitur, has duas series inter se esse æquales, neque enim necesse est, ut valores ipsius y^* inde orti fiant æquales, dummodo quantitate constante a se invicem differant.*" And in page 109 and 110, he has calculated the difference of the two series

* This y answers to z in the following problems.

derived from the same fluxionary equation. But since, in some cases, the quantity sought would not be very accurately determined by the method there used, he has left to those who come after him room for the exercise of their ingenuity, in devising others which shall better answer the purpose.

Of the difference of the values given by the two kinds of series of which I have been speaking, there is, in page 6 of Sir ISAAC NEWTON's *Analysıs per quantitatum series*, &c. printed in 1711, an instance too plain to have escaped the notice of that great man; and in page 51 he mentions that the areas given by these series are different: but there is nothing said of a constant difference. For what reasons he has not treated of that difference, and given rules for computing it, I shall not take upon me to enquire; but had he so done, there had been no room left for this Essay, nor, in all probability, for any other Treatise on the subject.

I now proceed to the solution of some problems, in which the difference that has been spoken of will be pointed out and computed.

PROBLEM I.

Given $\dot{z} = \dot{x} \sqrt{a+x}$, where x and z begin together, to find the correct value of z in series which shall converge when x is greater than a .

In

In order to treat this problem with the greater perspicuity, I will set down the values of z obtained by different methods of redintegration; and then, by comparing one of those values with another, their equality or difference will appear.

By a well known method of redintegration, the correct equation of the fluents, in finite terms, is

$$(1) \quad z = \frac{2}{3} \cdot \overline{a+x}^{\frac{3}{2}} - \frac{2}{3} a^{\frac{3}{2}}.$$

But if the quantity affected by the radical sign, in the given equation, be converted into an infinite series, in which a is made the leading term, and the fluents be taken, we shall have

$$(2) \quad z = x \sqrt{a} \times : 1 + \frac{x}{2 \cdot 2a} - \frac{x^2}{2 \cdot 4 \cdot 3a^2} + \frac{1 \cdot 3x^3}{2 \cdot 4 \cdot 6 \cdot 4a^3} - \frac{1 \cdot 3 \cdot 5x^4}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 5a^4}, \&c.$$

which series, it is evident, will converge as long as x is not greater than a .

But if $\sqrt{x+a}$, the radical quantity in the given equation, be converted into an infinite series, in which x is made the leading quantity, and the fluents be taken, there will be

$$(3) \quad z = \sqrt{x} \times : \frac{2}{3} x + a + \frac{a^2}{4x} - \frac{a^3}{4 \cdot 6x^2} + \frac{3a^4}{4 \cdot 6 \cdot 8x^3}, \&c.$$

which series will converge as long as x is not less than a .

Now, before the values of z given in the first and second equations can be compared with each other, the radical quantity in the former must be reduced into

into series, in which a is made the leading quantity.
This operation may stand as below.

$$\sqrt{a+x} \text{ is } = \sqrt{a} \times \left\{ 1 + \frac{x}{2a} - \frac{x^2}{2 \cdot 4 a^2} + \frac{1 \cdot 3 x^3}{2 \cdot 4 \cdot 6 a^3} - \frac{1 \cdot 3 \cdot 5 x^4}{2 \cdot 4 \cdot 6 \cdot 8 a^4}, \&c. \right.$$

$a + x$

and

$$\frac{a+x}{a+x} \sqrt{a+x} = \left\{ \begin{array}{l} \sqrt{a} \times \left\{ a + \frac{x}{2} - \frac{x^2}{2 \cdot 4 a} + \frac{1 \cdot 3 x^3}{2 \cdot 4 \cdot 6 a^2} - \frac{1 \cdot 3 \cdot 5 x^4}{2 \cdot 4 \cdot 6 \cdot 8 a^3}, \&c. \right. \\ \left. + x + \frac{x^2}{2a} - \frac{x^3}{2 \cdot 4 a^2} + \frac{1 \cdot 3 x^4}{2 \cdot 4 \cdot 6 a^3}, \&c. \right. \end{array} \right.$$

$$= \sqrt{a} \times \left\{ a + \frac{3x}{2} + \frac{3x^2}{2 \cdot 4 a} - \frac{3x^3}{2 \cdot 4 \cdot 6 a^2} + \frac{3 \cdot 3 x^4}{2 \cdot 4 \cdot 6 \cdot 8 a^3}, \&c. \right.$$

If now $a\sqrt{a}$ be taken from both sides of this equation, and the remainder be divided by $\frac{2}{3}$, we shall have

$$\frac{2}{3} \cdot \frac{a+x}{a+x} \sqrt{a+x} - \frac{2}{3} a \sqrt{a} = \sqrt{a} \times \left\{ x + \frac{x^2}{4a} - \frac{x^3}{4 \cdot 6 a^2} + \frac{3x^4}{4 \cdot 6 \cdot 8 a^3}, \&c. \right.$$

which series perfectly agrees with that obtained above, in the second equation.

Let us now compare together the values of x given in the first and third equations, by the same method.

$$\frac{x+a}{x+a} \sqrt{x+a} = \left\{ \begin{array}{l} \sqrt{x} \times \left\{ 1 + \frac{a}{2x} - \frac{a^2}{2 \cdot 4 x^2} + \frac{1 \cdot 3 a^3}{2 \cdot 4 \cdot 6 x^3} - \frac{1 \cdot 3 \cdot 5 a^4}{2 \cdot 4 \cdot 6 \cdot 8 x^4}, \&c. \right. \\ \left. \times \frac{x+a}{x+a} \right. \end{array} \right.$$

$$= \left\{ \begin{array}{l} \sqrt{x} \times \left\{ x + \frac{a}{2} - \frac{a^2}{2 \cdot 4 x} + \frac{1 \cdot 3 a^3}{2 \cdot 4 \cdot 6 x^2} - \frac{1 \cdot 3 \cdot 5 a^4}{2 \cdot 4 \cdot 6 \cdot 8 x^3}, \&c. \right. \\ \left. + a + \frac{a^2}{2x} - \frac{a^3}{2 \cdot 4 x^2} + \frac{1 \cdot 3 a^4}{2 \cdot 4 \cdot 6 x^3}, \&c. \right. \end{array} \right.$$

$$= \sqrt{x} \times \left\{ x + \frac{3a}{2} + \frac{3a^2}{2 \cdot 4 x} - \frac{3a^3}{2 \cdot 4 \cdot 6 x^2} + \frac{3 \cdot 3 a^4}{2 \cdot 4 \cdot 6 \cdot 8 x^3}, \&c. \right.$$

Here

Here again, $a\sqrt{a}$ being taken from both sides of the equation, and the remainders divided by $\frac{3}{2}$, we have

$$\frac{3}{2}x + a\sqrt{x+a} - \frac{2}{3}a\sqrt{a} = \begin{cases} \sqrt{x} \times \frac{2}{3}x + a + \frac{a^2}{4x} - \frac{a^3}{4 \cdot 6x^2} + \frac{3a^4}{4 \cdot 6 \cdot 8x^3}, \&c. \\ -\frac{2}{3}a\sqrt{a}. \end{cases}$$

Now, since the quantities on the second side of this equation give the correct value of z , and differ from the series in the third equation by the constant quantity $\frac{2}{3}a\sqrt{a}$, it is evident that the series there set down requires a correction, being always too great by the quantity $\frac{2}{3}a\sqrt{a}$.

Here it may be proper to observe, that this correction is attainable also by a comparison of the series in the second and third equations. For x being put $= a$,

$$\left. \begin{array}{l} \text{The ascend. ser.} \\ \text{The descend. ser.} \end{array} \right\} \text{becomes} \begin{cases} a\sqrt{a} \times \frac{2}{3} + 1 + \frac{1}{4} - \frac{1}{4 \cdot 6} + \frac{3}{4 \cdot 6 \cdot 8} - \frac{3 \cdot 5}{4 \cdot 6 \cdot 8 \cdot 10}, \&c. \\ a\sqrt{a} \times \frac{2}{3} + 1 + \frac{1}{4} - \frac{1}{4 \cdot 6} + \frac{3}{4 \cdot 6 \cdot 8} - \frac{3 \cdot 5}{4 \cdot 6 \cdot 8 \cdot 10}, \&c. \end{cases}$$

where the latter evidently exceeds the former by the quantity $\frac{2}{3}a\sqrt{a}$.

Let us now illustrate this matter by some examples in numbers. First, let $a = 10$, and $x = 1$; then the value of z given by the fluent in finite terms, is $\frac{22}{3}\sqrt{11} - \frac{20}{3}\sqrt{10} = 3.24007$ nearly. By the ascending series

$$z = \sqrt{}$$

$$z = \sqrt{10} \times : 1 + \frac{1}{40} - \frac{1}{4.600} + \frac{1}{4.4.4000} - \frac{1}{4.4.80000}, \&c.$$

$$= 3.24007 \text{ nearly.}$$

For a second example, let $a=1$, and $x=10$; then, by the first equation of the fluents, $z = \frac{22}{3} \sqrt{11} - \frac{2}{3} \sqrt{1}$
 $= 23.65525$ nearly. By the descending series $z = -\frac{2}{3}$
 $+ \sqrt{10} \times : \frac{20}{3} + 1 + \frac{1}{40} - \frac{1}{4.600} + \frac{1}{4.4.4000} - \frac{1}{4.4.80000}, \&c.$
 $= 23.65525$ nearly.

PROBLEM II.

Let the equation proposed be $\dot{z} = \frac{\dot{x}}{a+x}$, where x and z begin together, to find the correct value of z in descending series.

The fraction on the second side of the equation being converted into series, in which x is the leading term, and the fluents taken, we have $z = \text{hyp. log. } x,$
 $+ \frac{a}{x} - \frac{a^2}{2x^2} + \frac{a^3}{3x^3} - \frac{a^4}{4x^4}, \&c.$

Now, to find whether or not this value of z needs any correction, we must have recourse to the ascending series, by which the correct value of z is expressed thus:

$$z = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}, \&c.$$

Now

Now put $x = a$, and

$$\left. \begin{array}{l} \text{The ascend. ser.} \\ \text{The descend. ser.} \end{array} \right\} \text{becomes} \left\{ \begin{array}{l} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}, \&c. \\ h.l. a, + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}, \&c. \end{array} \right.$$

where it is evident that the latter is too great by the quantity $h.l. a$. Therefore the correct equation of the fluents, in descending series, is $z = h.l. \frac{x}{a} + \frac{a}{x} - \frac{a^2}{2x^2} + \frac{a^3}{3x^3}, \&c.$

EXAMPLE. Let $a = 2$, and $x = 4$; then will $z = h.l. 2, + \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 8} - \frac{1}{4 \cdot 16}$, which is $= h.l. 2 + h.l. \frac{1}{2} = h.l. 3$. And this value perfectly agrees with the fluent taken in other terms, it being known that z is $= h.l. \frac{a+x}{a}$, which, when $a = 2$ and $x = 4$, becomes $h.l. \frac{2+4}{2}$, or 3.

PROBLEM III.

Let it be required to find the correct value of z in descending series, in this equation, $z = \frac{aax}{aa+xx}$, where x and z begin together.

By operations similar to those performed in the solution of the last problem, the following expressions of the fluent are obtained.

P

$z = x$

$$z = x \times : 1 - \frac{x^2}{3a^2} + \frac{x^4}{5a^4} - \frac{x^6}{7a^6}, \&c.$$

$$z = \frac{aa}{x} \times : -1 + \frac{a^2}{3x^2} - \frac{a^4}{5x^4} + \frac{a^6}{7x^6}, \&c.$$

The ascending series, it is evident, will vanish with z , and therefore needs no correction. And when $x = a$

$$\text{the ascend. ser. is } = a \times : 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}, \&c.$$

$$\text{the descend. ser. is } = a \times : -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7}, \&c.$$

$$\text{and their differ. } = 2a \times : 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}, \&c.$$

which is well known to be = circular arc of 90° whose radius is a . Call this difference d ; then shall the correct fluent required be

$$z = d - \frac{aa}{x} \times : 1 - \frac{a^2}{3x^2} + \frac{a^4}{5x^4} - \frac{a^6}{7x^6}, \&c.$$

EXAMPLE. Let $a = 1$, and $x = \sqrt{3}$; then will $z = a - \frac{1}{\sqrt{3}} \times : 1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 9} - \frac{1}{7 \cdot 27}, \&c. = \text{arc of } 60^\circ.$

PROBLEM IV.

Let the equation given be $\dot{z} = \dot{x} \sqrt{aa + xa}$, where x and z begin together, to find the correct value of z in descending series.

By the common methods of redintegration we have

$$z = ax$$

$$z = ax + \frac{x^3}{2.3a} - \frac{x^5}{2.4.5a^3} + \frac{3x^7}{2.4.6.7a^5} - \frac{3.5x^9}{2.4.6.8.9a^7}, \&c.$$

$$z = \frac{1}{2}x^3 + \frac{1}{2}aa \text{ h.l. } x, + \frac{a^4}{2.4.2x^2} - \frac{3a^6}{2.4.6.4x^4} + \frac{3.5a^8}{2.4.6.8.6x^6}, \&c.$$

The ascending series, it is evident, needs no correction; and x being put $= a$ these series become

$$aa \times : 1 + \frac{1}{2.3} - \frac{1}{2.4.5} + \frac{3}{2.4.6.7} - \frac{3.5}{2.4.6.8.9}, \&c.$$

$$\text{and } aa \times : \frac{1}{2} + \frac{1}{2} \text{ h.l. } a + \frac{1}{2.4.2} - \frac{3}{2.4.6.4} + \frac{3.5}{2.4.6.8.6}, \&c.$$

whose difference is

$$aa \times : \frac{1}{2} - \frac{1}{2} \text{ h.l. } a + \left\{ \begin{array}{l} \frac{1}{2.3} - \frac{1}{2.4.5} + \frac{3}{2.4.6.7} - \frac{3.5}{2.4.6.8.9}, \&c. \\ - \frac{1}{2.4.2} + \frac{3}{2.4.6.4} - \frac{3.5}{2.4.6.8.6}, \&c. \end{array} \right.$$

which, without much difficulty, will be found $= aa \times 0.5965, \&c. - \frac{1}{2} \text{ h.l. } a.$

But as the same correction may be computed by a different and easier method, it seems proper to give that also.

The correct equation of the fluents is

$$z = \frac{1}{2}x\sqrt{xx+aa} + \frac{1}{2}aa \text{ h.l. } \frac{x+\sqrt{xx+aa}}{a}$$

Now since $\sqrt{xx+aa}$, making x the leading quantity, is

$$= x + \frac{aa}{2x} - \frac{a^4}{2.4x^3}, \&c. \quad \frac{1}{2}x\sqrt{xx+aa} \text{ will be } = \frac{1}{2}x^2 + \frac{aa}{4} - \frac{a^4}{44x^2},$$

$$\&c. \text{ and } \frac{1}{2}aa \text{ h.l. } \frac{x+\sqrt{xx+aa}}{a} \text{ will be } = \frac{1}{2}aa \times : \text{ h.l. }$$

$$\frac{2x}{a}, + \frac{aa}{4x^2} - \frac{3a^4}{4.8x^4}, \&c. \quad \text{The sum of these two expressions}$$

sions when x is immensely great, is barely $\frac{1}{2}xx + \frac{aa}{4} + \frac{1}{2}aa \text{ h.l. } \frac{2x}{a}$; and the descending series, in that case, is $= \frac{1}{2}x^2 * + \frac{1}{2}aa \text{ h.l. } x$. The difference d , is $= \frac{aa}{4} + \frac{1}{2}aa \text{ h.l. } \frac{2}{a} = aa \times 0.5965736 - \frac{1}{2} \text{ h.l. } a$, which is the constant quantity to be added to the descending series,

P R O B L E M V.

Given $\dot{z} = \frac{x}{\sqrt{xx - aa}}$, where $z = 0$ when $x = a$, to find the correct value of z in descending series.

By a process similar to that in the solution of the preceding problem, we get $z = \text{h.l. } x, - \frac{a^2}{2.2x^2} - \frac{3a^4}{2.4.4x^4} - \frac{3.5a^6}{2.4.6.6x^6}, \&c.$ But the correct value of z is $\text{h.l. } \frac{x + \sqrt{xx - aa}}{a}$, which, when x is immensely great, becomes barely $\text{h.l. } \frac{2x}{a}$. The series, in that case, becomes barely $\text{h.l. } x$. The difference, therefore, $\text{h.l. } \frac{2}{a}$, being added to the series, gives $z = \text{h.l. } \frac{2x}{a}, - \frac{a^2}{2.2x^2} - \frac{3a^4}{2.4.4x^4} - \frac{3.5a^6}{2.4.6.6x^6}, \&c.$ the value which was to be found.

SCHOLIUM. When $x = a$, one side of the last equation becomes 0, and by transposition, we have
h. l. 2

h.l. $z = \frac{1}{2.2} + \frac{3}{2.4.4} + \frac{3.5}{2.4.6.6}$, &c. which, upon actual calculation, will be found to be true. And this is the value of M. EULER'S C, in page 100 of the first Vol. of his *Institutiones*; which value, however, is not there set down.

P R O B L E M VI.

Given $yy + ay + xy - aa - 2xx = 0$, and $y\dot{x} = \dot{z}$, to find the correct value of z in descending series.

By operations similar to those which have been performed above, the variable part of the fluent is found to be

$$\frac{1}{2}x^2 - \frac{ax}{3} + \frac{11}{3.9}aa \text{ h.l. } x, + \frac{11a^3}{3.9^2x} + \frac{110a^4}{6.9^3x^2} - \frac{352a^5}{9^4x^3}, \text{ \&c.}$$

of which all the terms after the third will vanish, when x is immensely great.

But the correct value of z , in other terms, is known to be

$$-\frac{1}{2}ax - \frac{1}{4}xx + \frac{a}{12} + \frac{3}{4}x \cdot \sqrt{\frac{5}{9}aa + \frac{2}{9}ax + xx} \\ + \frac{11}{27}aa \times \text{h.l.} \frac{a}{9} + x + \sqrt{\frac{5}{9}aa + \frac{2}{9}ax + xx} + c; \text{ in which } c \text{ is } = \\ -\frac{aa}{36} \sqrt{5} - \frac{11}{27}aa \times \text{h.l.} a \times \frac{1}{9} + \frac{1}{3} \sqrt{5} = aa \times 0.00101 - \frac{11}{27} \text{h.l. } a.$$

And this value of z , when x is immensely great, becomes

$$\frac{1}{2}x^2 -$$

$\frac{1}{2}x^2 - \frac{1}{3}ax + \frac{23}{108}aa + \frac{11}{27}aa \times \text{h.l. } 2x, + c$; from which
 subtract $\frac{1}{2}x^2 - \frac{1}{3}ax + \frac{11}{27}aa \times \text{h.l. } x$, the value of the
 series in that case, and there remains

$$\frac{23}{108}aa + \frac{11}{27}aa \times \text{h.l. } 2, + c; \text{ which re-}$$

mainder is easily reduced to $aa \times 0.496366 - \frac{11}{27}\text{h.l. } a$. Call
 it d ; then will the equation required be

$$z = d + \frac{1}{2}xx - \frac{1}{3}ax + \frac{11}{27}aa \times \text{h.l. } x + \frac{11a^3}{3 \cdot 9^2x} + \frac{110a^4}{6 \cdot 9^3x^2} - \frac{352a^5}{9^4x^3}, \&c.$$

SCHOLIUM. It will require no great skill in algebra
 to discover, that the variable part of the fluent is
 also $= \frac{1}{2}x^2 - \frac{1}{3}ax + \frac{11}{27}aa \times \text{h.l. } x + \frac{a}{9} + \frac{121a^4}{6 \cdot 9^3x^2} - \frac{121a^5}{3 \cdot 9^4x^3}, \&c.$
 which expression has some little advantage over the
 other.

P R O B L E M VII.

Given $y^3 + axy + x^2y - a^3 - 2x^3 = 0$, and $y\dot{x} = \dot{z}$, to
 find the correct value of z in descending series.

The variable part of the fluent here sought, as com-
 puted by Sir Isaac Newton, (see his tract *De analysi per*
æquationes infinitas, page 12) is

$$\frac{1}{2}xx - \frac{1}{4}ax + \frac{1}{64}aa \text{ h. l. } x - \frac{131a^3}{512x} - \frac{509a^4}{32768x^2} +, \&c.$$

And by different calculations I find the invariable
 part

part of it to be $aa \times 0.91077 - \frac{1}{64} h.l.a$; the sum of which quantities is the correct value of z .

S C H O L I U M.

The veneration I have for Sir *Isaac Newton*, (and which, surely, is due from all men of inferior abilities to his wonderfully penetrating and comprehensive mind,) made me very diffident in my own calculations, when I first applied myself to the solution of this problem. I was not easily convinced, that any calculation made by him could be corrected by any other man. After a time, however, several methods occurred to me of proving the defect of the fluent as it stands in the tract above quoted, one of which I will here subjoin.

$$\text{Since } y = x - \frac{a}{4} + \frac{aa}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3}, \&c.$$

$$y\dot{x} \text{ will be } = \dot{x} \times x - \frac{a}{4} + \frac{aa}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3}, \&c.$$

$$\text{and } x\dot{y} \text{ will be } = \dot{x} \times x * - \frac{aa}{64x} - \frac{2.131a^3}{512x^2} - \frac{3.509a^4}{16384x^3}, \&c.$$

and their sum

$$x\dot{y} + y\dot{x} = \dot{x} \times 2x - \frac{a}{4} * - \frac{131a^3}{512x^2} - \frac{2.509a^4}{16384x^3}, \&c.$$

the fluents of which give

$$xy = x^2 - \frac{ax}{4} * + \frac{131a^3}{512x} + \frac{509a^4}{16384x^2}, \&c.$$

But since the true value of y is expressed by the series
above

above written, the true value of xy must be

$$= x^2 - \frac{ax}{4} + \frac{aa}{64} + \frac{131a^3}{512x} + \frac{509a^4}{16384x^2}, \text{ \&c.}$$

which is evidently greater than the value of xy obtained by fluxions, by the constant quantity $\frac{aa}{4}$. This,

I say, proves that some correction is necessary; but what that correction ought to be must be determined by other methods. Here, then, a field is open for the evercise of ingenuity.

Why the correction of descending series was not sooner taught, and that kind of series brought into use, may, perhaps, be accounted for, if we consider the extent of the science of Fluxions, to which the words of *Seneca*, upon another occasion, are applicable: *Ad inquisitionem tantorum ætas una non sufficit.—Veniet tempus, quo ista quæ nunc latent, in lucem dies extrahat, et longioris ævi diligentia.*

E S S A Y VI.

On the Transformation of certain Series to others of Swifter Convergency.

THE series, whose transformations are here treated of, are some of the common ones that express logarithms and circular arcs; and the new forms, to which they are brought, are adapted to the purpose of numerical calculation, which end they are well fitted to answer, not only by converging much swifter than the original series, but by the simplicity and peculiar relation of their co-efficients.

Of the other advantages to be obtained from this method of transformation, some shall be mentioned towards the end of this paper.

§ I. The series $\frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} - \frac{1}{4a^4} + \frac{1}{5a^5} - \frac{1}{6a^6}, \&c.$ which is known to express the h. l. $\frac{a+1}{a}$, by collecting the 1st and 2d terms into one, the 3d and 4th terms into another, &c. is transformed to

$$\frac{2a-1}{2a^2} + \frac{4a-3}{3 \cdot 4a^4} + \frac{6a-5}{5 \cdot 6a^6} + \frac{8a-7}{7 \cdot 8a^8}, \&c.$$

which series is = the sum of these two

$$\frac{2a-2}{2a^2} + \frac{4a-4}{3 \cdot 4a^4} + \frac{6a-6}{5 \cdot 6a^6} + \frac{8a-8}{7 \cdot 8a^8}, \&c.$$

and $\frac{1}{2a^2} + \frac{1}{3 \cdot 4a^4} + \frac{1}{5 \cdot 6a^6} + \frac{1}{7 \cdot 8a^8}, \&c.$

Now if the numerators and denominators of the upper series be divided by 2, 4, 6, 8, &c. respectively, it will become

$$\frac{a-1}{a^2} + \frac{a-1}{3a^4} + \frac{a-1}{5a^6} + \frac{a-1}{7a^8}, \&c.$$

which is =

$$\frac{a-1}{a^2} \times : 1 + \frac{1}{3a^2} + \frac{1}{5a^4} + \frac{1}{7a^6}, \&c.$$

So that, L being put = the original series, we have

$$L = \left\{ \begin{array}{l} \frac{a-1}{a^2} \times : 1 + \frac{1}{3a^2} + \frac{1}{5a^4} + \frac{1}{7a^6}, \&c. \\ + \frac{1}{2aa} \times : 1 + \frac{1}{2 \cdot 3a^2} + \frac{1}{3 \cdot 5a^4} + \frac{1}{4 \cdot 7a^6}, \&c. \end{array} \right.$$

which two series not only converge twice as swiftly as the original one converges, but the terms of the one are derived from those of the other by easy divisions.

But the value of L may be expressed somewhat differently as follows:

$$L = \frac{2a-1}{2a^2} + \frac{4a-3}{3 \cdot 4a^4} + \frac{6a-5}{5 \cdot 6a^6} + \frac{8a-7}{7 \cdot 8a^8}, \&c.$$

is evidently

$$= \left\{ \begin{array}{l} \frac{a-1}{2a^2} + \frac{3a-3}{3 \cdot 4a^4} + \frac{5a-5}{5 \cdot 6a^6} + \frac{7a-7}{7 \cdot 8a^8}, \&c. \\ + \frac{a}{2a^2} + \frac{a}{3 \cdot 4a^4} + \frac{a}{5 \cdot 6a^6} + \frac{a}{7 \cdot 8a^8}, \&c. \end{array} \right.$$

from

from which, by operations similar to those which were performed above, we get

$$L = \left\{ \begin{array}{l} \frac{a-1}{2a^2} \times : 1 + \frac{1}{2a^2} + \frac{1}{3a^4} + \frac{1}{4a^6}, \&c. \\ + \frac{1}{2a} \times : 1 + \frac{1}{2 \cdot 3a^2} + \frac{1}{3 \cdot 5a^4} + \frac{1}{4 \cdot 7a^6}, \&c. \end{array} \right.$$

which form the value of L may be computed somewhat easier than by that above

EXAMPLE. Let $a = 2$; then, by the first form we have

$$\text{h. l. } \frac{1}{2} = \left\{ \begin{array}{l} \frac{1}{4} \times : 1 + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 16} + \frac{1}{7 \cdot 64}, \&c. \\ + \frac{1}{8} \times : 1 + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 5 \cdot 16} + \frac{1}{4 \cdot 7 \cdot 64}, \&c. \end{array} \right.$$

By the second form we have

$$\text{h. l. } \frac{3}{2} = \left\{ \begin{array}{l} \frac{3}{8} \times : 1 + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 16} + \frac{1}{4 \cdot 64}, \&c. \\ + \frac{1}{4} \times : 1 + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 5 \cdot 16} + \frac{1}{4 \cdot 7 \cdot 64}, \&c. \end{array} \right.$$

§ II. The series $\frac{1}{a} + \frac{1}{2a^2} + \frac{1}{3a^3} + \frac{1}{4a^4} + \frac{1}{5a^5}, \&c.$ which is $= \text{h. l. } \frac{a}{a-1}$, by collecting the first and second terms into one, the third and fourth into another, $\&c.$ is transformed to

$$\frac{2a+1}{2a^2} + \frac{4a+3}{3 \cdot 4a^4} + \frac{6a+5}{5 \cdot 6a^6} + \frac{8a+7}{7 \cdot 8a^8}, \&c.$$

which series is evidently $=$ the difference of these two,

$$\frac{2a+2}{2a^2} + \frac{4a+4}{3 \cdot 4a^4} + \frac{6a+6}{5 \cdot 6a^6} + \frac{8a+8}{7 \cdot 8a^8}, \&c.$$

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and

and $\frac{1}{2a^2} + \frac{1}{3 \cdot 4a^4} + \frac{1}{5 \cdot 6a^6} + \frac{1}{7 \cdot 8a^8}, \&c.$

so that, L being put for the sum of the original series, we shall, by operations similar to those performed in the preceding section, get

$$L = \begin{cases} \frac{a+1}{aa} \times : 1 + \frac{1}{3a^2} + \frac{1}{5a^4} + \frac{1}{7a^6}, \&c. \\ - \frac{1}{2aa} \times : 1 + \frac{1}{2 \cdot 3a^2} + \frac{1}{3 \cdot 5a^4} + \frac{1}{4 \cdot 7a^6}, \&c. \end{cases}$$

by which series the value of L may be much more expeditiously computed than by the original one.

But since $\frac{2a+1}{2a^2} + \frac{4a+3}{3 \cdot 4a^4} + \frac{6a+5}{5 \cdot 6a^6} + \frac{8a+7}{7 \cdot 8a^8}, \&c.$ is evidently

$$= \begin{cases} \frac{a+1}{2a^2} + \frac{3a+3}{3 \cdot 4a^4} + \frac{5a+5}{5 \cdot 6a^6} + \frac{7a+7}{7 \cdot 8a^8}, \&c. \\ + \frac{a}{2a^2} + \frac{a}{3 \cdot 4a^4} + \frac{a}{5 \cdot 6a^6} + \frac{a}{7 \cdot 8a^8}, \&c. \end{cases}$$

we shall, by proper reduction, obtain

$$L = \begin{cases} \frac{a+1}{2aa} \times : 1 + \frac{1}{2a^2} + \frac{1}{3a^4} + \frac{1}{4a^6}, \&c. \\ + \frac{1}{2a} \times : 1 + \frac{1}{2 \cdot 3a^2} + \frac{1}{3 \cdot 5a^4} + \frac{1}{4 \cdot 7a^6}, \&c. \end{cases}$$

which is another very convenient form for computing the numerical value of L .

EXAMPLE. Putting $a = 2$, we have by the first form, h. l. 2 =

$$L = \begin{cases} \frac{3}{4} \times : 1 + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 16} + \frac{1}{7 \cdot 64}, \&c. \\ - \frac{1}{8} \times : 1 + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 5 \cdot 16} + \frac{1}{4 \cdot 7 \cdot 64}, \&c. \end{cases}$$

By

By the second form we have

$$L = \begin{cases} \frac{3}{8} \times : 1 + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 16} + \frac{1}{4 \cdot 64}, & \&c. \\ -\frac{1}{4} \times : 1 + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 5 \cdot 16} + \frac{1}{4 \cdot 7 \cdot 64}, & \&c. \end{cases}$$

§ III. The sum of the series $\frac{1}{a} + \frac{1}{3a^3} + \frac{1}{5a^5} + \frac{1}{7a^7} + \frac{1}{9a^9}, \&c.$ is well known to be $= \frac{1}{2}$ h. l. $\frac{a+1}{a-1}$; call it L ; then, by collecting two terms into one, in the manner described above, we shall have

$$L = \frac{3a^2+1}{3a^3} + \frac{7a^2+5}{5 \cdot 7a^7} + \frac{11a^2+9}{9 \cdot 11a^{11}} + \frac{15a^2+13}{13 \cdot 15a^{15}}, \&c.$$

which series, it is evident, is = the difference of these two

$$\frac{4a^2+4}{3a^3} + \frac{8a^2+8}{5 \cdot 7a^7} + \frac{12a^2+12}{9 \cdot 11a^{11}} + \frac{16a^2+16}{13 \cdot 15a^{15}}, \&c.$$

$$\text{and } \frac{a^2+3}{3a^3} + \frac{a^2+3}{5 \cdot 7a^7} + \frac{a^2+3}{9 \cdot 11a^{11}} + \frac{a^2+3}{13 \cdot 15a^{15}}, \&c.$$

And the difference of these is evidently

$$= \begin{cases} \frac{4a^2+4}{a^3} \times : \frac{1}{3} + \frac{2}{5 \cdot 7a^4} + \frac{3}{9 \cdot 11a^8} + \frac{4}{13 \cdot 15a^{12}}, & \&c. \\ -\frac{a^2+3}{a^3} \times : \frac{1}{3} + \frac{1}{5 \cdot 7a^4} + \frac{1}{9 \cdot 11a^8} + \frac{1}{13 \cdot 15a^{12}}, & \&c. \end{cases}$$

by which series the value of L may be much more expeditiously computed than by the original one.*

But the value of L may be computed by other series, which converge by the same powers of a with the last, whose co-efficients are simpler. For

* These are the series used in the computation of the h. l. 2, in page 40.

$$\begin{aligned} & \frac{3a^2+1}{3a^3} + \frac{7a^2+5}{5.7a^7} + \frac{11a^2+9}{9.11a^{11}} + \frac{15a^2+13}{13.15a^{15}}, \&c. \text{ is} \\ = & \begin{cases} \frac{3a^2+3}{3a^3} + \frac{7a^2+7}{5.7a^7} + \frac{11a^2+11}{9.11a^{11}} + \frac{15a^2+15}{13.15a^{15}}, \&c. \\ -\frac{2}{3a^3} - \frac{2}{5.7a^7} - \frac{2}{9.11a^{11}} - \frac{2}{13.15a^{15}}, \&c. \end{cases} \end{aligned}$$

which is easily reduced to

$$\begin{cases} \frac{aa+1}{a^3} \times : 1 + \frac{1}{5a^4} + \frac{1}{9a^8} + \frac{1}{13a^{12}}, \&c. \\ -\frac{2}{a^3} \times : \frac{1}{3} + \frac{1}{5.7a^4} + \frac{1}{9.11a^8} + \frac{1}{13.15a^{12}}, \&c. \end{cases}$$

by which pair of series the numerical value of L may be more easily calculated than by the pair first given for that purpose.

But the value of L may be yet somewhat more easily computed by another pair of series, which is discovered thus:

$$\begin{aligned} & \frac{3a^2+1}{3a^3} + \frac{7a^2+5}{5.7a^7} + \frac{11a^2+9}{9.11a^{11}} + \frac{15a^2+13}{13.15a^{15}}, \&c. \text{ is} \\ = & \begin{cases} \frac{a^2+1}{3a^3} + \frac{5a^2+5}{5.7a^7} + \frac{9a^2+9}{9.11a^{11}} + \frac{13a^2+13}{13.15a^{15}}, \&c. \\ + \frac{2a^2}{3a^3} + \frac{2a^2}{5.7a^7} + \frac{2a^2}{9.11a^{11}} + \frac{2a^2}{13.15a^{15}}, \&c. \end{cases} \end{aligned}$$

from which pair of series, by reduction, we obtain

$$L = \begin{cases} \frac{aa+1}{a^3} \times : \frac{1}{3} + \frac{1}{7a^4} + \frac{1}{11a^8} + \frac{1}{15a^{12}}, \&c. \\ + \frac{2}{a} \times : \frac{1}{3} + \frac{1}{5.7a^4} + \frac{1}{9.11a^8} + \frac{1}{13.15a^{12}}, \&c. \end{cases}$$

We have now obtained three new forms of series for the purpose of computing the numerical value of L ,

L , each of which converges twice as fast as the original one.*

EXAMPLE. Let $a = 3$; then, by these new forms, we have $\frac{1}{2} h.l. \frac{3+1}{3-1}$, or 2, expressed these different ways:

$$\begin{aligned} 1. \quad \frac{1}{2} h.l. 2 &= \begin{cases} \frac{40}{27} \times \frac{1}{3} + \frac{2}{5 \cdot 7 \cdot 81} + \frac{3}{9 \cdot 11 \cdot 81^2} + \frac{4}{13 \cdot 15 \cdot 81^3}, & \&c. \\ -\frac{12}{27} \times \frac{1}{3} + \frac{1}{5 \cdot 7 \cdot 81} + \frac{1}{9 \cdot 11 \cdot 81^2} + \frac{1}{13 \cdot 15 \cdot 81^3}, & \&c. \end{cases} \\ 2. \quad \frac{1}{2} h.l. 2 &= \begin{cases} \frac{10}{27} \times 1 + \frac{1}{5 \cdot 81} + \frac{1}{9 \cdot 81^2} + \frac{1}{13 \cdot 81^3}, & \&c. \\ -\frac{2}{27} \times \frac{1}{3} + \frac{1}{5 \cdot 7 \cdot 81} + \frac{1}{9 \cdot 11 \cdot 81^2} + \frac{1}{13 \cdot 15 \cdot 81^3}, & \&c. \end{cases} \\ 3. \quad \frac{1}{2} h.l. 2 &= \begin{cases} \frac{10}{27} \times \frac{1}{3} + \frac{1}{7 \cdot 81} + \frac{1}{11 \cdot 81^2} + \frac{1}{15 \cdot 81^3}, & \&c. \\ +\frac{2}{3} \times \frac{1}{3} + \frac{1}{5 \cdot 7 \cdot 81} + \frac{1}{9 \cdot 11 \cdot 81^2} + \frac{1}{13 \cdot 15 \cdot 81^3}, & \&c. \end{cases} \end{aligned}$$

§ IV. The series $\frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5} - \frac{1}{7n^7} + \frac{1}{9n^9}, \&c.$ (A) which is known to be = the circular arc whose tangent is $\frac{1}{n}$, by collecting every two terms into one, in the manner that has been already described, is transformed to

$$\frac{3n^2-1}{3n^3} + \frac{7n^2-5}{5 \cdot 7n^7} + \frac{11n^2-9}{9 \cdot 11n^{11}} + \frac{15n^2-13}{13 \cdot 15n^{15}}, \&c.$$

from which, by operations similar to those which have been performed in the preceding sections, we obtain

* By comparing the series in page 23, 24, and 25, with what has been done in this Essay, the truth of what was said in page 33, will appear.

the following three forms for computing the numerical value of A , in each of which the series converge twice as fast as the original one.

$$\begin{aligned}
 1. \quad A &= \begin{cases} \frac{4n^2-4}{n^3} \times : \frac{1}{3} + \frac{2}{5.7n^4} + \frac{3}{9.11n^8} + \frac{4}{13.15n^{12}}, & \&c. \\ -\frac{n^2-3}{n^3} \times : \frac{1}{3} + \frac{1}{5.7n^4} + \frac{1}{9.11n^8} + \frac{1}{13.15n^{12}}, & \&c. \end{cases} \\
 2. \quad A &= \begin{cases} \frac{n^2-1}{n^3} \times : 1 + \frac{1}{5n^4} + \frac{1}{9n^8} + \frac{1}{13n^{12}}, & \&c. \\ +\frac{2}{n^3} \times : \frac{1}{3} + \frac{1}{5.7n^4} + \frac{1}{9.11n^8} + \frac{1}{13.15n^{12}}, & \&c. \end{cases} \\
 3. \quad A &= \begin{cases} \frac{n^2-1}{n^3} \times : \frac{1}{3} + \frac{1}{7n^4} + \frac{1}{11n^8} + \frac{1}{15n^{12}}, & \&c. \\ +\frac{2}{n} \times : \frac{1}{3} + \frac{1}{5.7n^4} + \frac{1}{9.11n^8} + \frac{1}{13.15n^{12}}, & \&c. \end{cases}
 \end{aligned}$$

Let us now illustrate these series by some examples in numbers.

EXAMPLE I. When $n=1$, each of these three forms of series gives $A = 2 \times : \frac{1}{3} + \frac{1}{5.7} + \frac{1}{9.11} + \frac{1}{13.15}$, &c. the arc of 45° .

EXAMPLE II. Let n be $= \sqrt{3}$; then we shall have these three different expressions of the value of A , the arc of 30° .

$$\begin{aligned}
 1. \quad A &= \begin{cases} \frac{8}{3\sqrt{3}} \times : \frac{1}{3} + \frac{2}{5.7.9} + \frac{3}{9.11.9^2} + \frac{4}{13.15.9^3}, & \&c. \\ * & * & * & * \end{cases} \\
 2. \quad A &= \begin{cases} \frac{2}{3\sqrt{3}} \times : 1 + \frac{1}{5.9} + \frac{1}{9.9^2} + \frac{1}{13.9^3}, & \&c. \\ +\frac{2}{3\sqrt{3}} \times : \frac{1}{3} + \frac{1}{5.7.9} + \frac{1}{9.11.9^2} + \frac{1}{13.15.9^3}, & \&c. \end{cases}
 \end{aligned}$$

$$A = \begin{cases} \frac{2}{3\sqrt{3}} \times \frac{1}{3} + \frac{1}{7 \cdot 9} + \frac{1}{11 \cdot 9^2} + \frac{1}{15 \cdot 9^3}, \&c. \\ + \frac{2}{\sqrt{3}} \times \frac{1}{3} + \frac{1}{5 \cdot 7 \cdot 9} + \frac{1}{9 \cdot 11 \cdot 9^2} + \frac{1}{13 \cdot 15 \cdot 9^3}, \&c. \end{cases}$$

In this case it is remarkable that one of the two series in the first form vanishes, so that, at first sight, it might seem that the calculation of the value of A by that form would be easier than by either of the subsequent ones; but upon trial it will be found otherwise. And since, when n has any of those other values which are still more eligible for the purpose of computing the ratio of the diameter to the circumference of the circle, the calculations by the 2d and 3d forms will be easier than by the 1st, I shall make no further use of it.

EXAMPLE III. Let $n = 2$, and (for the sake of distinction in a subsequent use of it,) let the arc whose tangent is $\frac{1}{2}$, be $= \alpha$; then, by the second form we have

$$\alpha = \begin{cases} \frac{3}{8} \times \frac{1}{3} + \frac{1}{5 \cdot 16} + \frac{1}{9 \cdot 16^2} + \frac{1}{13 \cdot 16^3}, \&c. \\ + \frac{2}{8} \times \frac{1}{3} + \frac{1}{5 \cdot 7 \cdot 16} + \frac{1}{9 \cdot 11 \cdot 16^2} + \frac{1}{13 \cdot 15 \cdot 16^3}, \&c. \end{cases}$$

By the third form

$$\alpha = \begin{cases} \frac{3}{8} \times \frac{1}{3} + \frac{1}{7 \cdot 16} + \frac{1}{11 \cdot 16^2} + \frac{1}{15 \cdot 16^3}, \&c. \\ + 1 \times \frac{1}{3} + \frac{1}{5 \cdot 7 \cdot 16} + \frac{1}{9 \cdot 11 \cdot 16^2} + \frac{1}{13 \cdot 15 \cdot 16^3}, \&c. \end{cases}$$

By either of which forms the value of α may quickly be computed to great exactness.

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EXAMPLE

EXAMPLE IV. Let $n = 3$, and put $\beta =$ the arc whose tangent is $\frac{1}{3}$; then, by the second form we have

$$\beta = \left\{ \begin{array}{l} \frac{8}{27} \times \frac{1}{3} + \frac{1}{5 \cdot 81} + \frac{1}{9 \cdot 81^2} + \frac{1}{13 \cdot 81^3}, \&c. \\ + \frac{2}{27} \times \frac{1}{3} + \frac{1}{5 \cdot 7 \cdot 81} + \frac{1}{9 \cdot 11 \cdot 81^2} + \frac{1}{13 \cdot 15 \cdot 81^3}, \&c. \end{array} \right.$$

By the third form

$$\beta = \left\{ \begin{array}{l} \frac{8}{27} \times \frac{1}{3} + \frac{1}{7 \cdot 81} + \frac{1}{11 \cdot 81} + \frac{1}{15 \cdot 81^3}, \&c. \\ + \frac{2}{3} \times \frac{1}{3} + \frac{1}{5 \cdot 7 \cdot 81} + \frac{1}{9 \cdot 11 \cdot 81^2} + \frac{1}{13 \cdot 15 \cdot 81^3}, \&c. \end{array} \right.$$

which series converge swiftly.

Now, since it is known that the sum of the two arcs whose tangents are $\frac{1}{2}$ and $\frac{1}{3}$ is = the arc of 45° , we have $\alpha + \beta =$ arc of 45° , and $4\alpha + 4\beta =$ the circumference of the circle whose diameter is 1.

And by writing for $n = 7, \frac{11}{2}, \frac{79}{3}$, and many other values, series of much quicker convergency may be obtained for the purpose of computing the ratio of the diameter to the circumference of the circle. The method of finding those values, indeed, belongs not to this place; but it is probable that I shall treat of it at some future opportunity.

§ V. All the above series were obtained by collecting two contiguous terms into one; but that is merely arbitrary; and by collecting other equidistant terms into one, other series of quicker convergency may yet

yet be obtained. Thus, by collecting into one the 1st and 3d, 2d and 4th, &c. terms of the series $\frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5} - \frac{1}{7n^7} + \frac{1}{9n^9} - \frac{1}{11n^{11}} + \frac{1}{13n^{13}}, \&c.$ we get these two series

$$\frac{5n^4+1}{5n^5} + \frac{13n^4+9}{9 \cdot 13n^{13}} + \frac{21n^4+17}{17 \cdot 21n^{21}} + \frac{29n^4+25}{25 \cdot 29n^{29}}, \&c.$$

and $-\frac{7n^4+3}{3 \cdot 7n^7} - \frac{15n^4+11}{11 \cdot 15n^{15}} - \frac{23n^4+19}{19 \cdot 23n^{23}} - \frac{31n^4+27}{27 \cdot 31n^{31}}, \&c.$

from which, by a process similar to those which have been set down in the former part of this paper, we obtain A , the arc whose tangent is $\frac{1}{n}$,

$$= \begin{cases} + \left\{ \begin{aligned} &\frac{n^4+1}{n^5} \times : \frac{1}{5} + \frac{1}{13n^8} + \frac{1}{21n^{16}} + \frac{1}{29n^{24}}, \&c. \\ &+ \frac{4}{n} \times : \frac{1}{5} + \frac{1}{9 \cdot 13n^8} + \frac{1}{17 \cdot 21n^{16}} + \frac{1}{25 \cdot 29n^{24}}, \&c. \end{aligned} \right. \\ - \left\{ \begin{aligned} &\frac{n^4+1}{n^7} \times : \frac{1}{7} + \frac{1}{15n^8} + \frac{1}{23n^{16}} + \frac{1}{31n^{24}}, \&c. \\ &+ \frac{4}{n^3} \times : \frac{1}{3 \cdot 7} + \frac{1}{11 \cdot 15n^8} + \frac{1}{19 \cdot 23n^{16}} + \frac{1}{27 \cdot 31n^{24}}, \&c. \end{aligned} \right. \end{cases}$$

EXAMPLE. Taking $n=\sqrt{3}$, we have A , the arc of 30° ,

$$= \begin{cases} + \left\{ \begin{aligned} &\frac{10}{9\sqrt{3}} \times : \frac{1}{5} + \frac{1}{13 \cdot 81} + \frac{1}{21 \cdot 81^2} + \frac{1}{29 \cdot 81^3}, \&c. \\ &+ \frac{4}{\sqrt{3}} \times : \frac{1}{5} + \frac{1}{9 \cdot 13 \cdot 81} + \frac{1}{17 \cdot 21 \cdot 81^2} + \frac{1}{25 \cdot 29 \cdot 81^3}, \&c. \end{aligned} \right. \\ - \left\{ \begin{aligned} &\frac{10}{27\sqrt{3}} \times : \frac{1}{7} + \frac{1}{15 \cdot 81} + \frac{1}{23 \cdot 81^2} + \frac{1}{31 \cdot 81^3}, \&c. \\ &+ \frac{4}{3\sqrt{3}} \times : \frac{1}{3 \cdot 7} + \frac{1}{11 \cdot 15 \cdot 81} + \frac{1}{19 \cdot 23 \cdot 81^2} + \frac{1}{27 \cdot 31 \cdot 81^3}, \&c. \end{aligned} \right. \end{cases}$$

In this example we see a swift convergency, although

though the original series converged but slowly. And by collecting other two equidistant terms into one, other series of still swifter convergency may be found.

§ VI. From what has been done above it appears, that the series obtained by this method of transformation always come out by pairs, of such peculiar relation to each other, that the terms of the one being computed, those of the other are most easily derived from them; the advantage of which, in numerical calculations, is very obvious.

It is evident also, that the method of transformation which has been now explained, is applicable to all series of which the terms are any geometrical progression whatever divided by any arithmetical progression; and that a general theorem may easily be made by any one who understands what has been done in this Essay.

It may be proper to add, that these series admit of various other transformations, all which may have their particular uses; but to treat of them here is not only foreign from the design of this Essay, but would swell the book much beyond its proper limits. I shall therefore point out only one advantage more to be derived from this method, and that is nearer approximations to the values of the original series than those which have been commonly given.

For

For instance, taking the second form of series which expresses the length of a circular arc, (see § IV.) and putting $\frac{1}{n} = t$, the tangent of A , we have

$$A = \begin{cases} \overline{t-t^3} \times ; 1 + \frac{1}{5} t^4 + \frac{1}{9} t^8, \&c. \\ + 2t^3 \times ; \frac{1}{3} + \frac{1}{5.7} t^4 + \frac{1}{9.11} t^8, \&c. \end{cases}$$

from which, by the method explained in *Simpson's Dissertation*, page 100, we obtain

$$A = \overline{t-t^3} \times \frac{45-16t^4}{45-25t^4} + t^3 \times \frac{6930-1856t^4}{10395-3675t^4} \text{ very nearly.}$$

Which approximation takes in the just value of the first six terms of the original series.

And in like manner may be found more accurate approximations to the values of logarithmic series than any other, that I know of, which have yet been published.

E S S A Y VII.

On the Force of oscillating Bodies on their Centers of Suspension.

AMONG the several Treatises of Mechanics that have come to my hands, there is none that treats of the force with which pendulums, and oscillating bodies, act on their centers of motion; which force, however, is not unworthy of attention, since, upon enquiry, it will be found to be, in some cases, several times greater than the weight of the oscillating body.

This subject was first offered to my consideration while I was in the Royal Observatory, at Greenwich, when some new clocks* were set up there, to be used in the Astronomical Observations. It had been found by experiment, that Clocks kept time better when they were in strong cases firmly fixed against a wall, than when they were in common cases slightly fixed to the wainscot of a room, or only resting on a floor. And as it was part of my business to keep an account of the going of these new clocks, whose pendulums

* These clocks were made by Mr. *John Arnold*, of London.

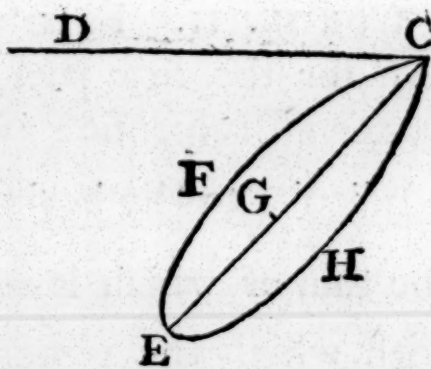
were

were much heavier than those of common clocks, I was induced to enquire with what force the pendulums acted on their centers of suspension, in an horizontal direction. That force, it was easy to perceive, would arise from two causes, the weight, and the centrifugal force, of the pendulum. The enquiry, therefore, led to the consideration of the following propositions:

PROPOSITION I.

Let C be the center of motion of the oscillating body CFEH, CE its axis, and W its weight; to find the effect of gravity on C, in an horizontal direction, or, with what force the point C is acted on in the direction CD, parallel to the horizon, in any position of the axis CE

Let G be the center of gravity of the oscillating body, the sine of the angle $ECD = c$, and its co-sine $= s$; then, by mechanics, the force with which the body acts on C, in the direction CE, will be $= cW$, the effect of which in the direction CD,



will be $= scW$. But this is evidently but part of the effect sought. For, whatever be the velocity of G, the center of gravity, the accelerating force of gravity

vity on it, in the direction perpendicular to CE, will always be $= sW$; the effect of which on C, in a direction perpendicular to CE, will be $= sW \times \frac{EG}{EC}$; but in the direction DC, it will be $= scW \times \frac{EG}{EC}$. Now, since this force acts always in a contrary direction to that above found, we have $scW \times 1 - \frac{EG}{EC}$ = the force sought.

COROL. I. The expression now obtained, of the horizontal force arising from gravity only, by writing $\frac{CG}{CE}$ for its equal, $\frac{CE-EG}{CE}$, becomes $scW \times \frac{CG}{CE}$; and since s and c only are variable quantities, it is evident that this force will be the greatest when the rectangle sc is greatest, which is well known to be when the angle ECD is $= 45^\circ$.

COROL. II. From what was done above, we have ccW for the first part of the force with which the center of suspension is acted on in a direction perpendicular to the horizon, and $ssW \times \frac{EG}{EC}$ for the other; the sum of which is $ccW + ssW \times \frac{EG}{EC}$ = the whole force with which the center C is acted on in a direction perpendicular to the horizon.

PROPOSITION

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But, before I proceed to investigation, it will be proper to premise, that the oscillating body, whose forces I shall consider in this paper, is supposed to be homogeneous, and formed by the revolution of a line about the axis, or else of such shape, that the ends of the solids into which it is cut, by the vertical plane in which its axis vibrates, by another plane, perpendicular to the former, in which the axis lies, and by a third plane, to which the axis is perpendicular, are similar and equal. The reasons for this choice will appear further on.

PROPOSITION I.

Let C be the center of motion of the oscillating body CFEH, CE its axis, and W its weight; to find the effect of gravity on C, in an horizontal direction, or, with what force the point C is acted on in the direction CD, parallel to the horizon, in any position of the axis CE.

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Let

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COROL. II. From what was done above, we have ccW for the first part of the force with which the center of suspension is acted on in a direction perpendicular to the horizon, and $ssW \times \frac{EG}{EC}$ for the other; the sum of which is $ccW + ssW \times \frac{EG}{EC}$ = the whole force with which the center C is acted on in a direction perpendicular to the horizon.

PROPOSITION

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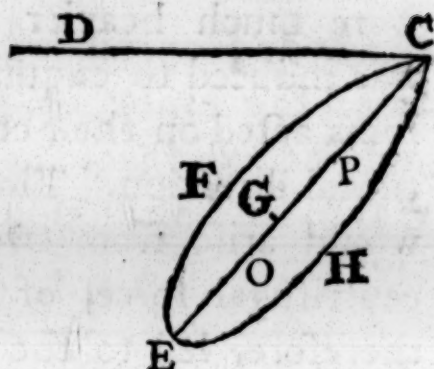
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S

Let

Let G be the center of gravity, and O the center of oscillation of the body, the sine of the angle $ECD = c$, and its co-sine $= s$; then, by mechanics, the force with which the body acts on C , in the direction CE , will be $= cW$, the effect of which in the direction CD , will be $= scW$.



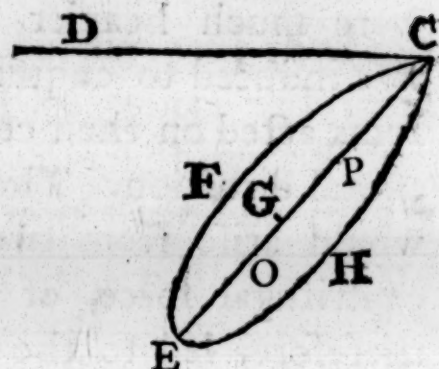
But this is evidently but part of the effect sought. For, whatever be the velocity of G , there will be an accelerating force of gravity upon it, in a direction perpendicular to CE , the effect of which on C may be thus computed. Let there be a section of the body, by a plane to which the axis is perpendicular, at p , which (the body being such as was described in page 127,) will be the center of gravity of the section. Let the weight of the particles in that section be w ; then will the force with which w acts on p , in a direction perpendicular to CE , be $sw \times \frac{Op}{OC}$, the effect of which on C , in the same direction, is $= sw \times \frac{Op^2}{OC^2}$. But $Op^2 = OC^2 - 2OC \times Cp + Cp^2$; therefore, we have $\frac{sw}{OC^2} \times OC^2 \times w - 2OC \times Cp \times w + Cp^2 \times w$, a general expression of the effect on C , in the direction perpendicular to CE , of any section of the

the body perpendicular to its axis, at the distance Cp from the center of suspension; from which, by the well-known theorems for computing the centers of gravity and oscillation of bodies, we obtain the force of all the sections, or of the whole body, $= \frac{s}{OC^2} \times OC^2 \times W - 2OC \times CG \times W + OC \times CG \times W = sW \times 1 - \frac{CG}{CO}$; which force, in the direction DC , will be $= scW \times 1 - \frac{CG}{CO}$. Now, since this force acts always in a contrary direction to that above found, we have $scW - scW \times 1 - \frac{CG}{CO} = scW \times \frac{CG}{CO}$, the force sought.

COROL. I. Since s and c only are variable quantities in the expression now obtained, it is evident that this force will be greatest when the rectangle sc is greatest, which is well known to be when the angle ECD is $= 45^\circ$.

COROL. II. From what was done above, we have ccW for the first part of the force with which the center of suspension is acted on in a direction perpendicular to the horizon, and $ssW \times 1 - \frac{CG}{CO}$ for the other; the sum of which is $W - ssW \times \frac{CG}{CO}$, the whole force with which the center C is acted on in a direction perpendicular to the horizon.

Let G be the center of gravity, and O the center of oscillation of the body, the sine of the angle $ECD = c$, and its co-sine $= s$; then, by mechanics, the force with which the body acts on C , in the direction CE , will be $= cW$, the effect of which in the direction CD , will be $= scW$.



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 $OC^2 \times W - 2OC \times CG \times W + OC \times CG \times W = sW \times 1 - \frac{CG}{CO}$;
 which force, in the direction DC , will be $= scW \times$
 $1 - \frac{CG}{CO}$. Now, since this force acts always in a contrary direction to that above found, we have
 $scW - scW \times 1 - \frac{CG}{CO} = scW \times \frac{CG}{CO}$, the force sought.

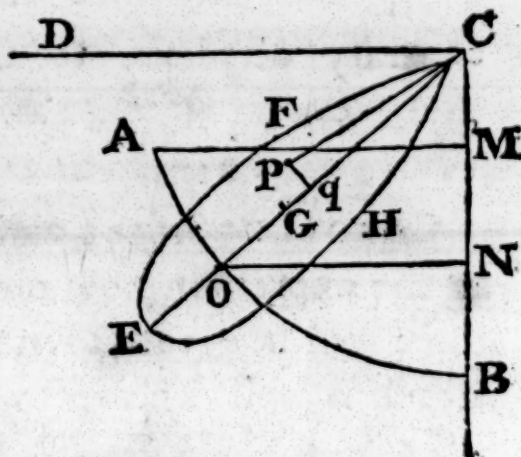
COROL. I. Since s and c only are variable quantities in the expression now obtained, it is evident that this force will be greatest when the rectangle sc is greatest, which is well known to be when the angle ECD is $= 45^\circ$.

COROL. II. From what was done above, we have ccW for the first part of the force with which the center of suspension is acted on in a direction perpendicular to the horizon, and $ssW \times 1 - \frac{CG}{CO}$ for the other; the sum of which is $W - ssW \times \frac{CG}{CO}$, the whole force with which the center C is acted on in a direction perpendicular to the horizon.

PROPOSITION II.

Let C be the center of suspension of the oscillating body $CFEH$, W its weight, COE the axis, and AOB half the arc described by O , the center of oscillation; it is required to find with what force C is urged in the direction CD , parallel to the horizon, by the centrifugal force of the body.

Let $CFEH$ be a section of the body by the vertical plane in which the axis CE vibrates; let the velocity of O in feet, per second, be V , the sine of the angle $OCB = s$, and its co-sine $= c$; let p be any point in the section, and draw pq perpendicular to CE ; then will the cen-



trifugal force of a particle at O be as $\frac{VV}{CO}$, which put $= F$. Now, since the angular velocity of p (and every other point in the body,) is the same as that of O , the square of the velocity of p , per second, will be $= VV \times \left(\frac{Cp}{CO}\right)^2$, and the centrifugal force of a particle at p , in the direction Cp , will be as $\frac{VV}{CO^2} \times Cp$, which is to its force in the direction of the axis as Cp to Cq ; therefore

therefore its centrifugal force will be as $\frac{VV}{CO^2} \times Cq =$
 $\frac{VV}{CO} \times \frac{Cq}{CO} = F \times \frac{Cq}{CO}$, which, it is evident, is to the centri-
 fugal force of an equal particle at O, as Cq to CO.
 And by a similar argument it will appear, that the
 force, in direction of CE, of every other particle in
 the transverse section of the oscillating body by a plane
 passing through p and q, and perpendicular to the
 plane of the section CFEH, will be as $F \times \frac{Cq}{CO}$;
 and consequently, that the force of all the particles
 in that section will be as $F \times \frac{Cq}{CO} \times \text{area of the section}$;
 and the force of the whole body as $\frac{F}{CO} \times \text{sum of all the}$
particles in the body $\times$ *the distances of their sections from*
C: i. e. G being the center of gravity of the oscil-
 lating body, its centrifugal force will be as $F \times \frac{CG}{CO} \times$
the content of the body. Now, to compare this force with
 the weight of the body, draw AM and ON perpen-
 dicular to CB, and put $f = 32\frac{1}{6}$; then will V, the
 velocity of the center of oscillation, be $= \sqrt{MN \times 2f}$,
 and $\frac{VV}{CO} (=F) = \frac{MN \times 2f}{CO}$, which force is to the force
 of gravity as $\frac{2MN}{CO}$ to 1; and since the content of the
 body and its weight are in direct proportion to each
 other, we have $W \times \frac{2MN \times CG}{CO^2} =$ the centrifugal force
 of the body in the direction of its axis, the effect of
 which in the direction CD, is $= sW \times \frac{2MN \times CG}{CO^2}$, the
 force required.

COROL.

COROL. Since the centrifugal force of the body, in direction of its axis, is $= W \times \frac{2MN \times CG}{CO^2}$, in the direction of CB, it will be $= cW \times \frac{2MN \times CG}{CO^2}$.

SCHOLIUM. Here it will be proper to note, that the expressions now obtained will not be accurate, unless the centrifugal force of the particles of the body be equal on all sides of the axis; which it will be when the body is homogeneous and formed by the revolution of any line whatever about the axis, or of such other shape as is described in page 127.

PROPOSITION III.

The same things being given as in the last proposition, it is required to find the position of the axis CE when the whole force of the body on C, in an horizontal direction, is greatest. (See the figure to Prop. II.)

By the two preceding propositions, the whole force with which the oscillating body acts on C, in an horizontal direction, is universally $= scW \times \frac{CG}{CO} + sW \times \frac{2MN \times CG}{CO^2}$, from which it will be easy to find the position required, either by Algebra or a Geometrical Construction, the latter of which is to be preferred for a reason that will presently appear.* The

* Although the Geometrical construction is to be preferred for the demonstration of the limit of the Maximum, yet, as the Algebraic process

The values of s and c are evidently $\frac{NO}{CO}$ and $\frac{CN}{CO}$ respectively, which being written for s and c in the above

process affords an easy method of computing the required position of the axis, it seems proper to give that also.

Putting $\frac{CM}{CO} = d$, we have $\frac{2MN}{CO} \left(= \frac{2CN - 2CM}{CO} \right) = 2c - 2d$, and the above general expression becomes $= \frac{CG}{CO} W \times \overline{3sc - 2sd}$, the variable part of which is under the vinculum; and by putting its fluxion $= 0$, and proper reduction, we have $cc - \frac{d}{3}c = \frac{1}{2}$, and $c = \frac{d + \sqrt{18 + dd}}{6}$.

COROL. I. When the arc AB is a quadrant, $d = 0$, and we have $c = \frac{\sqrt{2}}{2}$, which is the co-sine of 45° .

COROL. II. When the arc AB is a semicircle, $d = -1$, and $c = \frac{-1 + \sqrt{19}}{6}$.

COROL. III. When $d = c$, the equation $cc - \frac{d}{3}c = \frac{1}{2}$, becomes $cc - \frac{cc}{3} = \frac{1}{2}$, from which we get $c = \frac{\sqrt{3}}{2}$, the co-sine of 30° , which is the limit of the maximum.

SCHOLIUM. Since, when the arc AB is greater than a quadrant, the sign of d must be changed, and, in that case, c (in the general expression above) passes through all degrees of magnitude from $-d$ to $+1$; it is evident that there must be a point at which $-3c + 2d$ will be $= 0$, at which the horizontal force will also be $= 0$; and consequently, while c has any value from $-d$ to $-\frac{2d}{3}$, the horizontal force will be negative, i. e. it will act in a contrary direction.

expression

COROL. I. When the arc AB is a quadrant, $CM=0$ and the horizontal force is greatest when the angle $OCB=45^\circ$; in which case the force is expressed by $\frac{3CG}{2CO}W$.

COROL. II. When the arc AB is greater than a quadrant, both the point M and the line PQ will ascend above CD, and CM must have the sign— before it.

COROL. III. When the arc AB is greater than a quadrant, and MC is to MN as 3 to 1, the horizontal force will be $=0$; and consequently, when MC is to MN in a greater ratio than that of 3 to 1, the horizontal force will be negative, or it will act in a contrary direction.

SCHOLIUM. From the above construction it appears, that the maximum is restricted to a certain limit, beyond which the greatest horizontal force will always be at A, the end of the arc. For, since the less the arc AB is, the greater will CM, and consequently MP be, it is evident that AB may be taken so small, that ON may coincide with AM, or even ascend above it. The limit evidently is when ON coincides with AM, and is thus determined. Since NS is $=$ NP when the rectangle OP is a maximum, it follows, that, when ON coincides with AM, NS will be $=$ MP; and, by the construction,

T

NS

NS (=MP) will be to NC (=MC) as 1 to 3, which is also the ratio of the tangent of OB to its co-tangent: from which the angle OCB is very easily found = 30° .

P R O P O S I T I O N IV.

The same things being still given, it is required to find the position of the axis of the oscillating body when its force on the center of suspension, in a perpendicular direction, is greatest.

By Corol. 2 to proposition 1, that part of the force here sought which arises from gravity, is universally $W - ssW \times \frac{CG}{CO}$, which quantity, it is evident, will be greatest when $s=0$, i. e. when the axis of the body is perpendicular to the horizon, and then becomes barely $=W$.

By prop. 2, it appears, that the centrifugal force of the body is greatest when its velocity is greatest, which is at the bottom of the arc, when MN becomes =MB. Therefore, the position sought is a perpendicular to the horizon; in which position the whole force is universally $= W \times 1 + \frac{2MB \times CG}{CO^2}$, which, by putting $\frac{MB}{CO} = v$, becomes $= W \times 1 + \frac{2v \times CG}{CO}$.

It may now be proper to illustrate these propositions by an application of them to the solution of some problems.

P R O B-

P R O B L E M I.

To find the greatest horizontal force of a pendulum 20lb. weight, when half the arc of vibration is 2° , the distances of the centers of oscillation and gravity from the center of suspension being as 46 to 45.

In this case, the greatest horizontal force is at the end of the given arc, and arises from gravity only. We have, therefore, by Prop. 1, $0.0349 \times 0.9994 \times 20 \times \frac{45}{46} = 0.683\text{lb.}$ nearly, or 10.92 oz. the force sought.*

P R O B L E M II.

The same things being given, it is required to find the greatest perpendicular force with which the pendulum acts on the center of suspension.

By Prop. 4, we have $20 \times 1 + 0.00122 \times \frac{45}{46} = 20.0239\text{lb.}$
 $= 20\text{lb. } 0.38\text{ oz.}$ nearly, the force sought.

* This solution is to be considered as more exact than that in the *Ladies' Diary* for 1786, where no allowance was made for the difference of the length of the pendulum and the distance of its center of gravity from the center of suspension.

It is to be further remarked, that, in these solutions, the pendulum is supposed to be perfectly inflexible; but when there is a spring in the upper part of the pendulum, even this solution must be subject to some little inaccuracy: the error, however, cannot be estimated without experiment.

P R O B L E M III.

To find both the greatest horizontal and perpendicular force with which a cylinder of lead, 4 feet long and 2000lb. weight, suspended at one end of its axis, and vibrating in a semicircle, acts on the center of suspension.

By Corol. 1, page 135, we have $\frac{18}{16} \times 2000 = 2250$ lb. the greatest horizontal force. And by Prop. 4, $2000 \times 1 + \frac{12}{8} = 2000 \times \frac{5}{2} = 5000$ lb. is the greatest perpendicular force.

P R O B L E M IV.

Let the cylinder described in problem 3 vibrate a whole circle, and let it be required to find both the greatest horizontal and perpendicular force with which it acts on the center of suspension.

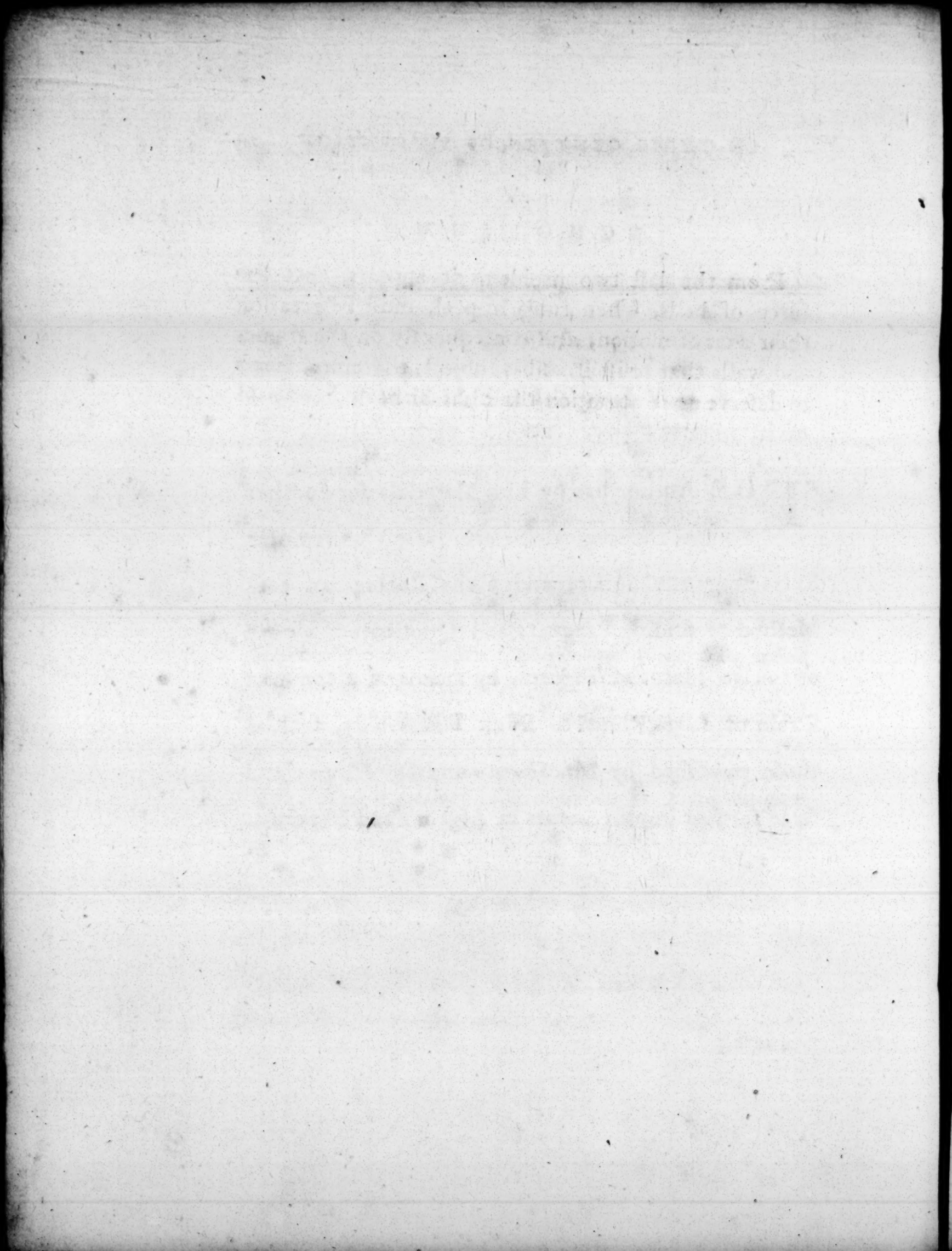
By Corol. 2, page 133, $c = 0.559816$, the co-sine of $55^\circ 57' 24''.8$, the sine of which angle is $= 0.828617$; and then, by the general expression in the note to Prop. 3, we have $\frac{3}{4} \times 2000 \times 3.679450 \times 0.828617 = 4573$ lb. nearly, the greatest horizontal force. By Prop. 4, we have $2000 \times 1 + \frac{24}{8} = 4 \times 2000 = 8000$ lb. the greatest perpendicular force.

S C H O-

S C H O L I U M.

From the last two problems it appears, that the force of bells, when rung in peal, must be great on their axes of motion, and consequently on the frames and walls that resist it. This subject, therefore, seems to deserve more attention than has hitherto been paid to it.

F I N I S.



A D V E R T I S E M E N T.

TH E Author has by him Materials for another Volume: among which are, New Theorems for Extracting the Square and Cube Roots; an easy Method of finding Products and Quotients to eleven or twelve Places of Figures, by means of a common Table of Logarithms to seven Places only, such as those published by Mr. *Sherwin* and Dr. *Hutton*; and some further improvements in Algebra and Fluxions.

CORRECTIONS to be made before the Book is read.

N. B. b. signifies count from the Bottom.

Page 5. To the Note at the bottom add, And Mr. Jones's *Synopsis Palmariorum Matheseos*.

Page 8. l. 12. for Equation, read *Equation*.

Page 19. for which perhaps is as small a number as need be used in computing a table of logarithms, after the modulus is found, read, which is about as small a number as need be used for this purpose in computing a table of logarithms.

Page 20. l. 11. for a , read a^2 .

Page 25. l. 2. b. for $3a$, read $\frac{3}{4}a$.

Page 32. l. 3. b. for page 27, read page 26.

Page 36. l. 1. for γ , read γ .

Page 39. l. 4. for $-$, read $=$.

Page 40. b. for following specimen, read table now to be computed.

Page 106. l. 7. b. for a , read d .

Page 112. l. 4. for $\frac{aa}{4}$ read $\frac{aa}{64}$. And l. 9. for *evercise*, read *exercise*.



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